The Ramanujan Machine: Automatically Generated Conjectures on Fundamental Constants

Anonymous Author(s)

Abstract

Fundamental mathematical constants like e and π are ubiquitous in diverse fields of science, from abstract mathematics and geometry to physics, biology and chemistry. Nevertheless, for centuries new mathematical formulas relating fundamental constants have been scarce and usually discovered sporadically. In this paper we propose a novel and systematic approach that leverages algorithms for deriving new mathematical formulas for fundamental constants and help reveal their underlying structure. Our algorithms find dozens of well-known as well as previously unknown continued fraction representations of π, e, and the Riemann zeta function values. Two new conjectures produced by our algorithm, along with many others, are:

$$e = 3 + \frac{-1}{4 + \frac{-2}{5 + \frac{-3}{6 + \frac{-4}{7 + \ldots}}}} \qquad , \qquad \frac{4}{\pi - 2} = 3 + \frac{1 \cdot 3}{5 + \frac{2 \cdot 4}{7 + \frac{3 \cdot 5}{9 + \frac{4 \cdot 6}{11 + \ldots}}}}$$

We present two algorithms that proved useful in finding new results: a variant of the Meet-In-The-Middle (MITM) algorithm and a Gradient Descent (GD) tailored to the recurrent structure of continued fractions. Both algorithms are based on matching numerical values and thus find new conjecture formulas without providing proofs and without requiring prior knowledge on any mathematical structure. This approach is especially attractive for fundamental constants for which no mathematical structure is known, as it reverses the conventional approach of sequential logic in formal proofs. Instead, our work presents a new conceptual approach for research: computer algorithms utilizing numerical data to unveil new internal structures and conjectures, thus playing the role of mathematical intuition of great mathematicians of the past, providing leads to new mathematical research.

Code available at: `http://www.ramanujanmachine.com/` or `https://github.com/AnonGit90210/RamanujanMachine`.

1 Introduction

Fundamental mathematical constants such as e, π, the golden ratio φ, and many others play an instrumental part in diverse fields such as geometry, number theory, calculus, fundamental physics, biology, and ecology [1]. Throughout history simple formulas of fundamental constants symbolized simplicity, aesthetics, and mathematical beauty. A couple of well-known examples include Euler's identity $e^{i\pi} + 1 = 0$ or the continued fraction representation of the Golden ratio:

$$\varphi = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}. \tag{1}$$

The discovery of such Regular Formulas (RFs) [1] was often sporadic and considered an act of mathematical ingenuity or profound intuition. One prominent example is Gauss' ability to see meaningful patterns in numerical data that led to new fields of analysis such as elliptic and modular functions and to the hypothesis of the Prime Number Theorem. He is even famous for saying "I have the result, but I do not yet know how to get it" [2], which emphasizes the role of identifying patterns and RFs in data as enabling acts of mathematical discovery.

In a different field but in a similar manner, Johannes Rydberg's discovery of his formula of hydrogen spectral lines [3], resulted from his data analysis of the spectral emission by chemical elements: $\lambda^{-1} = R_H(n_1^{-2} - n_2^{-2})$, where λ is the emission wavelength, R_H is the Rydberg constant, n_1 and n_2 are the upper and lower quantum energy levels respectively. This insight, emerging directly from identifying patterns in the data, had profound implications on modern physics and quantum mechanics.

Unlike measurements in physics and all other sciences, **mathematical** constants can be calculated to an arbitrary precision (number of digits) with an appropriate formula, thus providing an **absolute ground truth**. In this sense, mathematical constants contain an unlimited amount of data (e.g. the infinite sequence of digits in an irrational number), which we propose to use as a ground truth for finding new RFs. Since the fundamental constants are universal and ubiquitous in their applications, finding such patterns can reveal new mathematical structures with broad implications, e.g. the Rogers-Ramanujan continued fraction (which has implications on modular forms) and the Dedekind η and j functions [4, 5]. Consequently, having a **systematic** method to derive new RFs can help research in many fields of science.

In this paper, we establish a novel method to learn mathematical relations between constants and we present a list of new conjectures found using this method. While the method can be leveraged for many forms of RFs, we demonstrate its potential with equations of the form of generalized continued fractions (GCFs):

$$x = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}}, \tag{2}$$

where $a_n, b_n \in \mathbb{Z}$ for $n = 1, 2, \dots$ are partial numerators and denominators respectively. GCFs in which the partial numerators and denominators follow a closed-form expression like a polynomial have been of interest to mathematicians for centuries and still are today, e.g. William Broucker's π representation [6] or [1, 7, 8].

We demonstrate our approach by finding identities between a GCF and the value of a rational function at a fundamental constant. For simplicity, enumeration and expression aesthetics, we limit ourselves to integer polynomials on both sides of the equality. We propose two search algorithms. The first algorithm uses the Meet-In-The-Middle (MITM) algorithm to a relatively small precision in order to reduce the search space and eliminate mismatches. It increases the precision with a larger number of GCF iterations on the remaining hits to validate them as new conjectured RFs, and is therefore called MITM-RF. The second algorithm uses an optimization-based method, which we call Descent&Repel, converging to integer lattice points that define new conjectured RFs.

[1] By regular formulas we refer to any mathematical expression or equality that is infinite in nature but can be encapsulated using a finite expression.

Our MITM-RF algorithm was able to produce several novel conjectures, for example:

$$\frac{4}{\pi-2} = 3 + \cfrac{1\cdot 3}{5 + \cfrac{2\cdot 4}{6 + \cfrac{3\cdot 5}{7 + \cfrac{4\cdot 6}{8 + \cfrac{5\cdot 7}{9 + \cfrac{6\cdot 8}{10 + \cdots}}}}}}$$

$$e = 3 + \cfrac{-1}{4 + \cfrac{-2}{5 + \cfrac{-3}{6 + \cfrac{-4}{7 + \cfrac{-5}{8 + \cfrac{-6}{9 + \cdots}}}}}}$$

$$\frac{2}{\pi+2} = 0 + \cfrac{1\cdot(3-2\cdot 1)}{3 + \cfrac{2\cdot(3-2\cdot 2)}{6 + \cfrac{3\cdot(3-2\cdot 3)}{9 + \cfrac{4\cdot(3-2\cdot 4)}{12 + \cfrac{5\cdot(3-2\cdot 5)}{15 + \cdots}}}}}$$

$$e - 2 = 1 + \cfrac{2}{1 + \cfrac{-1}{1 + \cfrac{4}{1 + \cfrac{-2}{1 + \cfrac{6}{1 + \cfrac{-3}{1 + \cdots}}}}}}$$

Conjectures 1-4: Sample of automatically generated conjectures for mathematical formulas of fundamental constants, as generated by our proposed Ramanujan Machine by applying the MITM-RF algorithm. All these results are previously unknown conjectures to the best of our knowledge. Both results for π converge exponentially and both results for e converge super-exponentially. See Table 3 in the Appendix for additional results from our algorithms along with their convergence rates, which we separate to previously known formulas and new formulas.

One may wonder whether the conjectures discovered by this work are indeed mathematical identities or merely mathematical coincidences that breakdown once enough digits are calculated. However, the method employed in this work makes it fairly unlikely for the conjectures to breakdown. For an enumeration space of 10^9 and result accuracy of more than 50 digits, the probability of finding a random match is smaller than 10^{-40}. This minuscule probability makes us believe that the new conjectures are truths awaiting a rigorous proof by the mathematical community. In the past, the development of such proofs led to new discoveries, such as the consequences on number theory of the proof of Fermat's last theorem [9]. We believe and hope that proofs of these new conjectures will lead to new discoveries in the future.

After discovering dozens of GCFs we observed empirically that there is a relationship between the ratio of the polynomial order of a_n and b_n and the rate at which the formula converges as a function of the number of iterations. This relationship was also proven rigorously in the Appendix.

In contrast to the method we present, many known RFs of fundamental constants were discovered by conventional mathematical proofs, i.e. sequential logical steps derived from known properties of these constants. For example, several RFs of e and π were generated using the Taylor expansion of the exponent and the trigonometric functions and using Euler's continued fraction formula [10], which connects an infinite sum and an infinite GCF. In our work, we aim to reverse this process, finding new RFs for the fundamental constants using their numerical data alone, without any prior knowledge about their mathematical structure. Each RF may enable reverse-engineering of the mathematical structure that produces the RF, and provide new insight on the field. Our approach is especially powerful in cases of empirical constants, such as the Feigenbaum constant from chaos theory (Table 2), which are derived numerically from simulations and have no analytic representation.

Given the success of our approach in finding new RFs, there are many additional avenues for more advanced algorithms and future research. Inspired by worldwide collaborative efforts in mathematics such as the Great Internet Mersenne Prime Search (GIMPS) we launch the initiative `www.RamanujanMachine.com`, dedicated for finding new RFs for fundamental constants. The general community can donate computational time to find RFs, propose mathematical proofs for conjectured RFs, or suggest new algorithms for finding them (see Appendix Section 4).

2 Related Work

Stated in an oversimplified manner, the process of mathematical research usually includes two main steps: conjecturing and proving (as in Fig. 1). It is the latter step that was studied extensively in the computer science literature and is known as Automated Theorem Proving (ATP), which focuses on proving **existing conjectures** - fundamentally different from our work that focuses on generating **new conjectures**. In ATP, algorithms already proved many theorems [11], most notably the Four Color Theorem [12], the Lorenz attractor problem [13], the Keppler Conjecture on the density of sphere packing [14], as well as proving a conjectured identity for $\zeta(4)$ [15], and recent results [16, 17]. In contrast, it is the "conjecturing" step of the process of mathematical research that is the focus of this paper: automatic conjecturing of RFs.

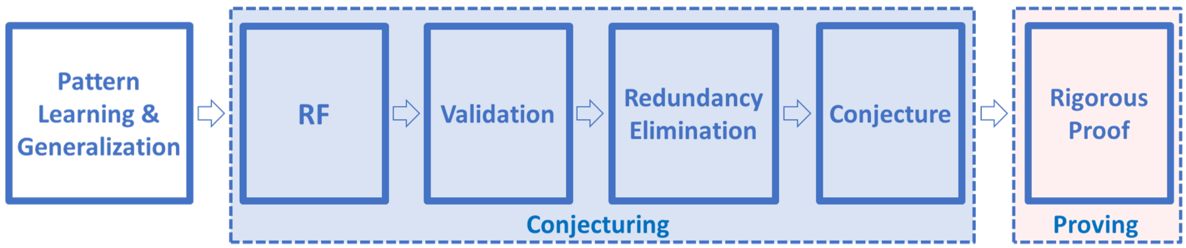

Figure 1: Conceptual flow of our Ramanujan Machine. First, through various approaches of pattern learning & generalization (Section 5.1), we can generate a space of RF conjectures, e.g. GCFs. We then apply a search algorithm, validate potential conjectures, and remove redundant results. Finally, the validated results form mathematical conjectures that need to be proven analytically, thus closing a complete research endeavour from pattern generation to a proof and potentially a new mathematical insight.

Proposing conjectures is often times more significant than proving them. For this reason some of the most original mathematicians and scientists are known for their famous **unsolved conjectures** rather than for their solutions to other problems, like Fermat's last theorem, Hilbert's problems, Landau's problems, Hardy-Littlewood prime tuple conjecture, Birch-Swinnerton-Dyer conjecture, and of course the Riemann Hypothesis [9, 18, 19, 20, 21]. Maybe the most famous example is Ramanujan, who posed dozens of conjectures involving fundamental constants and considered them to be revelations from one of his goddesses [22]. In our work, **we aim to automate the process of conjectures generation**. We demonstrate this concept by providing new conjectures for fundamental constants.

3 The Meet-In-The-Middle-RF Algorithm

Given a fundamental constant c (e.g. $c = \pi$), our goal is to learn a set of four polynomials (α,β,γ,δ):

$$\frac{\gamma(c)}{\delta(c)} = f_i\left(\mathrm{GCF}(\alpha, \beta)\right) \tag{3}$$

for $\{f_i\}$ a given set of functions (e.g. $f_1(x) = x,\ f_2(x) = \frac{1}{x}, \ldots$), where $\mathrm{GCF}(\alpha, \beta)$ means the generalized continued fraction with the partial numerator and denominator $a_n = \alpha(n)\,;\, b_n = \beta(n)$ respectively as defined in Eq. (2). α, β, γ and δ are integer polynomials.

As showcased in Fig. 2, we start by enumerating over the two sides of Eq. (3) and successively

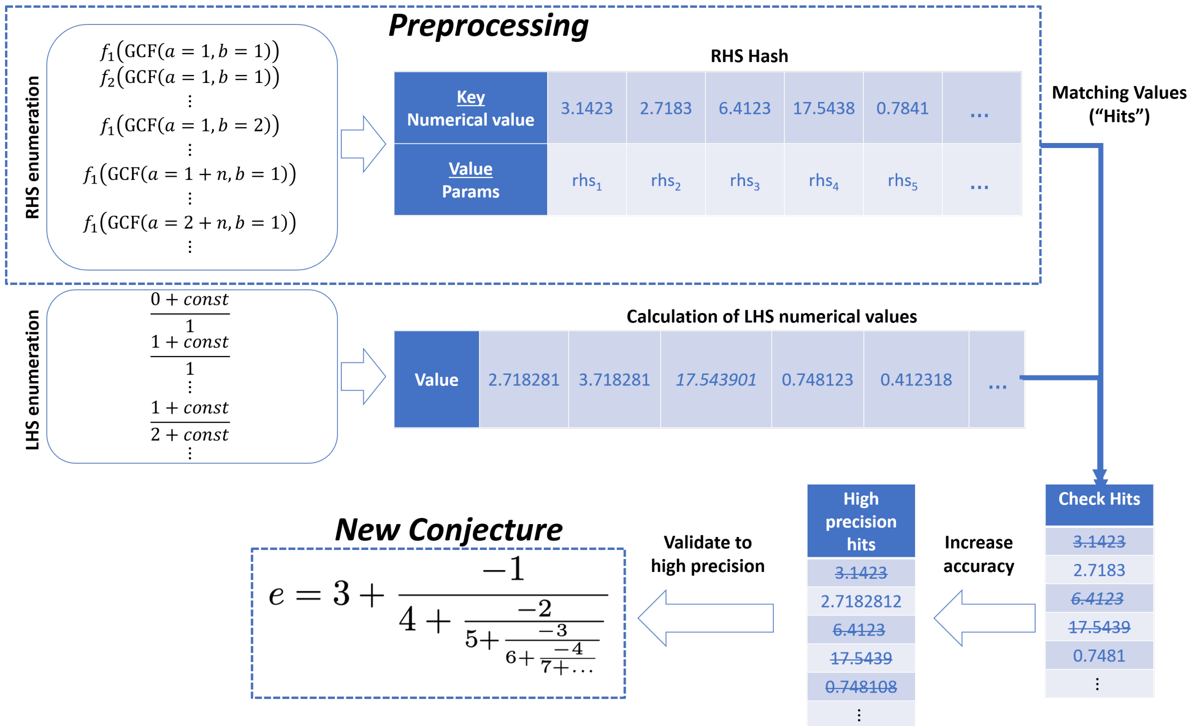

Figure 2: The Meet-In-The-Middle (MITM-RF) method: first we enumerate RHS to a low precision, values are stored in a hash-table. Then we enumerate LHS to a low precision and search for matches. The matches are reevaluated to a higher precision and compared again. The process is repeated until a specified, arbitrary decimal precision is reached, thus reducing false positives. The final results are then posed as new conjectures.

generating many different integer polynomials for $\alpha, \beta, \gamma, \delta$ [2]. We calculate the Right-Hand-Side (RHS) of each instance up to a limited number of iterations and store the results in a hash-table. We continue by evaluating the Left-Hand-Side (LHS) up to a pre-selected decimal point. We attempt to then locate each result from the LHS in the hash-table with the RHS results, where successful attempts are considered as candidate solutions, and will be referred to as "hits".

Since the LHS and RHS calculations are performed up to a limited precision, several of the hits are bound to be false positives. We then eliminate these false positives by calculating the rational function to an arbitrary precision to reduce the likelihood that the equality is coincidental as shown in Fig. 3.

A naive enumeration method is very computationally intensive with time complexity of $O(MN)$, where M and N are the LHS and RHS space size respectively, and space complexity of $O(1)$. Since calculating the RHS is more computationally costly, we store the RHS in the hash-table in order to significantly reduce computation time at the expense of space. This makes the algorithm's time complexity $O(M+N)$ and its space complexity $O(N)$. Moreover, the hash-table of the GCF (RHS) can be saved and reused for further LHS enumerations, reducing future enumeration durations by a significant amount.

We also generalize the aforementioned algorithm to allow for α and β to be interlaced sequences, i.e. they may consist of multiple integer polynomials. The most simple example of a non-trivial interlaced sequence is a sequence for which even values of n are equal to one polynomial and odd values of n are equal to a different polynomial. For results and details see Appendix Section 1 and the MITM-RF code on *www.RamanujanMachine.com*.

[2] We delete instances which produce trivial results like $\gamma = 3 \cdot \delta$ or $\alpha = 0$ and instances whose β polynomial has zero roots, which result in a finite GCF, necessarily representing a rational number.

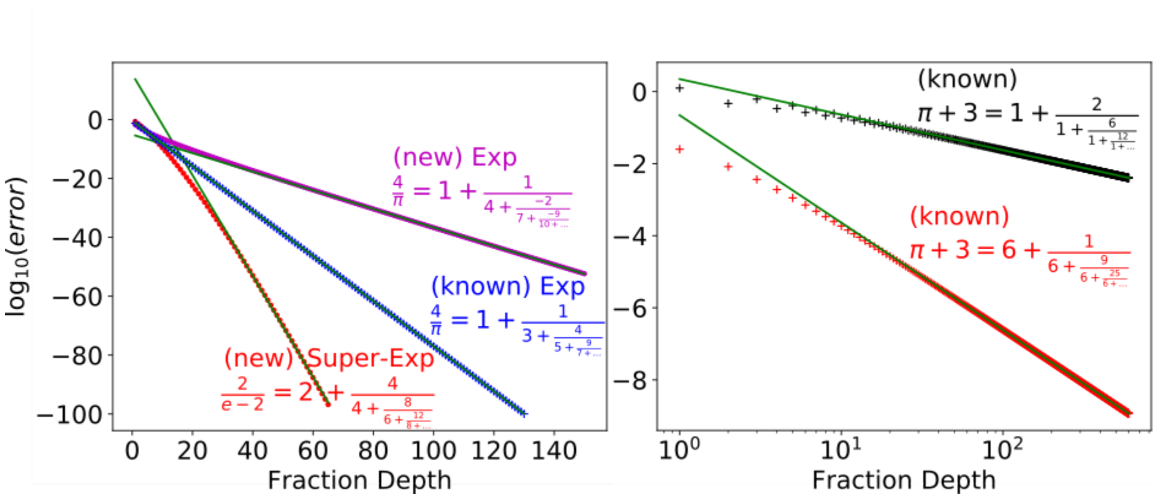

Figure 3: Convergence rate of the GCFs found by our method. Plots of the absolute difference between the GCFs and the corresponding fundamental constant (i.e. the error) vs. the depth of the GCF. All the results were found by our MITM-GCF algorithm. On the left are GCFs that converge exponentially/super-exponentially (validated numerically), and on the right are GCFs which converge polynomially. The vast majority of previously known results for π converge polynomially, while all of our newly found results converge exponentially or super-exponentially.

With our proposed MITM algorithm we were able to discover new regular GCFs for fundamental constants other than those previously known. However, seeing how successful our algorithm was despite being relatively simplistic, we believe there is still ample room for new results, which should follow by leveraging more sophisticated algorithms and more precise techniques, thus discovering hidden truths about fundamental constants that may be considered to be more exotic than π and e perhaps with formulas that are more complex than the GCFs that were used in this work.

3.1 Other Fundamental Constants with the MITM-RF

We also studied other fundamental constants of more exotic nature than π and e and found two new GCFs for Apéry's constant $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$. Note that the MITM-RF algorithm does not need to use any prior knowledge on the fundamental constant. However, there is a vast body of research on the properties of many fundamental constants from which various structures can be inferred. Hence when aiming for such a constant, one promising way to utilize such prior knowledge is to study other formulas of the fundamental constant, in attempt to find a common element and use that as a prior for the MITM-RF algorithm. Such an approach can reduce dramatically the enumeration space and the computational complexity, thus improving the chance for finding possible solutions. As a proof of concept of this approach for the Apéry constant, consider formulas 1 & 2.

$$\frac{1}{\zeta(3)} = 0^3 + 1^3 - \cfrac{1^6}{1^3 + 2^3 - \cfrac{2^6}{2^3 + 3^3 - \cfrac{3^6}{3^3 + 4^3 - \cfrac{4^6}{4^3 + 5^3 + \cdots}}}}$$

$$\frac{5}{2\zeta(3)} = 2 + 0 \cdot 2 \cdot 4 + \cfrac{2 \cdot 1^5 \cdot 1}{2 + 1 \cdot 3 \cdot 7 + \cfrac{2 \cdot 2^5 \cdot 3}{2 + 2 \cdot 4 \cdot 10 + \cfrac{2 \cdot 3^5 \cdot 5}{2 + 3 \cdot 5 \cdot 13 + \cfrac{2 \cdot 4^5 \cdot 7}{2 + 4 \cdot 6 \cdot 16 + \cdots}}}}$$

Formulas 1 & 2: New formulas for $\zeta(3)$ discovered using prior knowledge incorporated into MITM. The top formula converges polynomially while to bottom converges exponentially. See Appendix Section 2 and the attached code for details.

4 Descent&Repel

We propose a GD optimization method and demonstrate its success in finding RFs, and compare it with the MITM-RF method. The MITM-RF method, although proved successful, is not trivially scalable. This issue can be targeted by either a more sophisticated variant or by switching to an optimization based method, as is done by the following algorithm.

As explained in Section 3, we want to find integral solutions to Eq. (3). This can also be written as the following constrained optimization problem:

$$\begin{aligned} \underset{\alpha,\beta,\gamma,\delta}{\text{minimize}} \quad & \mathcal{L} = \left\| \frac{\gamma(\pi)}{\delta(\pi)} - \text{GCF}(\alpha,\beta) \right\|. \\ \text{subject to} \quad & \{\alpha,\beta,\gamma,\delta\} \ \subset \mathbb{Z}[x] \end{aligned} \tag{4}$$

Solving this optimization problem with GD appears implausible, since we are only satisfied with global minima without any error and the solutions must be integers. However, we found an important feature of the loss landscape of the described problem that allows it to be solved with a slightly modified GD that we name 'Descent&Repel' (Fig. 4, example of results in Table 1). The minima are not 0-dimensional points but $(d-1)$-dimensional manifolds with d being the number of optimization variables as would be expected given the single constraint. Moreover, we observed empirically that all minima are global and their errors are zero, therefore any GD process will result in a solution with $\mathcal{L} = 0$. It is well known that any real number can be expressed as a simple continued fraction [23], and the aforementioned feature hints that this may also be true for GCFs with integer polynomials.

Convergence	Known / New	Formula	Polynomials
Exponential	known	$\frac{4}{\pi} = 1 + \frac{1}{3+\frac{4}{5+\frac{9}{7+\dots}}}$	$a_n = 1 + 2n,\ b_n = n^2$
Super-Exponential	new	$e = 3 + \frac{-1}{4+\frac{-2}{5+\frac{-3}{6+\dots}}}$	$a_n = 3 + n,\ b_n = -n$

Table 1: RFs for π and e found in a proof-of-concept run of the Descent&Repel algorithm.

We chose the variables of the optimization problem as the coefficients of the $\alpha, \beta, \gamma, \delta$ polynomials in Eq.(4). The optimization problem is initiated with a large set of points, specifically in the examples we present all initial conditions were set on a line, as is showcased in Fig. 4. We iteratively perform GD for each point and then force all points to repel from one another via a "Coulomb"-like repulsion. To find integer solutions, we finalize our algorithm by GD steps toward the integer lattice and toward the minima curve, thus returning only solutions that lie on the integer lattice.

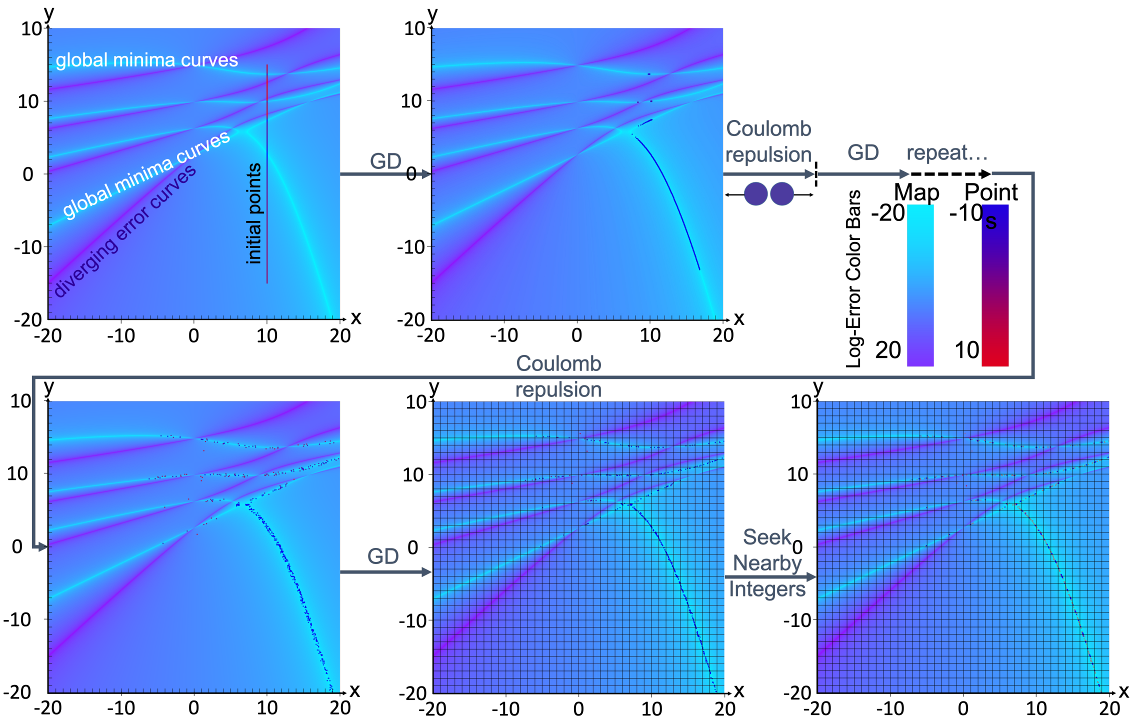

Figure 4: Schematic diagram of our Descent&Repel algorithm for finding new RFs for fundamental constants, by relying on GD optimization. The key observation that enables this method is that all minima are global ($\mathcal{L} = 0$) and appear as $(d-1)$-dimensional manifolds, where d is the number of optimization variables. Starting with our initial conditions (in this example, consisting of 600 points) on a line in (a), we perform ordinary GD alternating with "Coulomb" repulsion between all the points (b,c). Finally, to arrive to grid points, we perform GD toward integral points (with the loss function $\sin^2(\pi x)$) and toward the minimum curves, alternately (d). Lastly, we check whether any point satisfies the equation.

5 Discussion

5.1 Hypotheses Generation

Our results so far point to new interesting questions and hypotheses about fundamental constants: For example, we found many more continued fractions for e than for other constants we tested, despite the much smaller space tested for it with our algorithm. Why does it seem that some fundamental constants have more RFs compared to others? More generally, which fundamental constants can even be expressed with polynomial GCFs? Could there be constants (also in Section 4) for which RFs don't exist at all? It is intriguing that the novel research method we propose with the Ramanujan Machine not only finds new conjectures about RFs of fundamental constants, but also about the intrinsic mathematical structures.

A new conjecture about the mathematical structure of GCFs that emerged from this research and that we successfully proved concerns the rate of convergence of a GCF as a function of the degrees of the α, β polynomials. We observed and later proved that when $\frac{\deg(\beta)}{\deg(\alpha)} > 2$, then the convergence is always polynomial in the GCF depth. When the ratio is smaller than 2, then the convergence is always super-exponential. When the ratio is precisely 2, then the convergence can be exponential, depending on more subtle conditions (see Appendix Section 3 for details). This result allowed us to further improve MITM-RF algorithm.

We propose a systematic way of generating a space of candidate RF conjectures, generalizing beyond the examples that we explored above. To establish new candidate mathematical conjectures, we envision harvesting the scientific literature (e.g., arXiv.org containing over 1.5M papers) and generalizing RFs with machine learning algorithms such as clustering methods. The rich dataset available online should provide a strong ground truth for candidate RFs, which can be explored using algorithms similar to the ones described in this work. Such approach may discover many new mathematical conjectures that go far beyond GCFs and can be explored in a future work.

5.2 Applications

New RF conjectures could have intriguing applications. Fast converging GCFs and other identities are being utilized for efficient calculation of different constants, for example, one of the most efficient methods to compute π is based on a formula by Ramanujan [24]. More generally, new RFs could help us calculate other constants faster, like the super-exponential convergence that was demonstrated above for e. Another potential application of new RFs is for proving intrinsic properties of fundamental constants. An example is Apéry's proof that $\zeta(3)$ is irrational, done by representing it as a GCF [25], which led to similar proofs for other constants.

5.3 The Universality of Fundamental Constants

Field	Name	Decimal Expansion
Related to Continued Fractions	Lévy's constant	$\gamma = 3.275822\ldots$
	Khinchin's constant	$K_0 = 2.685452\ldots$
Physics	First Feigenbaum constant	$\delta = 4.669201\ldots$
	Second Feigenbaum constant	$\alpha = 2.502907\ldots$
	Laplace Limit	$\lambda = 0.662743\ldots$
Number Theory	Twin Prime constant	$\Pi_2 = 0.660161\ldots$
	Meissel – Mertens constant	$M = 0.261497\ldots$
	Landau–Ramanujan constant	$\Lambda = 0.764223\ldots$
Cominatorics	Euler–Mascheroni constant	$\gamma = 0.577215\ldots$
	Catalan's constant	$G = 0.915965\ldots$
...	...	...

Table 2: A sample of fundamental constants from different fields, which are all relevant targets for our method, a wider list is available in [1] and [5]. For all of these, new RF conjectures will point to deep underlying connections. There are thousands more constants for which enough numerical data exists and our method is applicable. With further improvement on our suggested approaches, along with new algorithms provided by the community, we expect that such expressions will be found. Note, that some constants in the table like the Feigenbaum constants have no analytical expression what-so-ever, and so far can only be computed using numerical simulation. Therefore, having a RF for them will reveal a hidden truth not only about the constant, but also about the entire field to which it relates.

We have so far only provided the groundwork for a far more comprehensive study into fundamental constants and their underlying mathematical structure. With our proposed algorithms and their extensions, we were able to find RFs for the constants π, e, and $\zeta(3)$. Table 2 presents a selection of additional fundamental constants of particular interest to our approach. For part of them, e.g. Feigenbaum constants,

no RF is known. We also list a few examples of constants with intrinsic connections to the theory of GCF. Potentially the most interesting constants for further research are the ones coming from other fields, like number theory (not so ironically, some of them are also named after Ramanujan) and various fields of physics. With such constants, any new RF can point to a new hidden connection between fields of science. With further improvements and new algorithms, applied on the thousands of fundamental constants in the literature, we expect many new RFs to be found.

References

[1] Steven R Finch and Jet Wimp. Reviews-mathematical constants. *Mathematical Intelligencer*, 26(2):70–73, 2004.

[2] Jonathan Borwein and David Bailey. *Mathematics by experiment: Plausible reasoning in the 21st century*. AK Peters/CRC Press, 2008.

[3] Niels Bohr. Rydberg's discovery of the spectral laws. 1954.

[4] Goro Shimura. Modular forms of half integral weight. In *Modular Functions of One Variable I*, pages 57–74. Springer, 1973.

[5] Wolfram. Mathworld, 2019.

[6] Joseph Frederick Scott. *The mathematical work of John Wallis (1616-1703)*. Taylor and Francis, 1938.

[7] Thomas J Pickett and Ann Coleman. Another continued fraction for π. *The American Mathematical Monthly*, 115(10):930–933, 2008.

[8] Dawei Lu, Lixin Song, and Yang Yu. Some new continued fraction approximation of euler's constant. *Journal of Number Theory*, 147:69–80, 2015.

[9] Andrew Wiles. Modular elliptic curves and fermat's last theorem. *Annals of mathematics*, 141(3):443–551, 1995.

[10] Leonhard Euler. *Introductio in analysin infinitorum*, volume 2. MM Bousquet, 1748.

[11] Marko Petkovšek, Herbert S Wilf, and Doron Zeilberger. A= b, ak peters ltd. *Wellesley, MA*, 30, 1996.

[12] Kenneth I Appel and Wolfgang Haken. *Every planar map is four colorable*, volume 98. American Mathematical Soc., 1989.

[13] Warwick Tucker. The lorenz attractor exists. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 328(12):1197–1202, 1999.

[14] Thomas C Hales. A proof of the kepler conjecture. *Annals of mathematics*, pages 1065–1185, 2005.

[15] David H Bailey, Jonathan M Borwein, and Roland Girgensohn. Experimental evaluation of euler sums. *Experimental Mathematics*, 3(1):17–30, 1994.

[16] William YC Chen, Qing-Hu Hou, and Doron Zeilberger. Automated discovery and proof of congruence theorems for partial sums of combinatorial sequences. *Journal of Difference Equations and Applications*, 22(6):780–788, 2016.

[17] Kshitij Bansal, Sarah M Loos, Markus N Rabe, Christian Szegedy, and Stewart Wilcox. Holist: An environment for machine learning of higher-order theorem proving (extended version). *arXiv preprint arXiv:1904.03241*, 2019.

[18] Steve Smale. Mathematical problems for the next century. *The mathematical intelligencer*, 20(2):7–15, 1998.

[19] Godfrey H Hardy, John E Littlewood, et al. Some problems of 'partitio numerorum'; iii: On the expression of a number as a sum of primes. *Acta Mathematica*, 44:1–70, 1923.

[20] John Tate. On the conjectures of birch and swinnerton-dyer and a geometric analog. *Séminaire Bourbaki*, 9(306):415–440, 1965.

[21] Edmund Landau. Vorlesungen über zahlentheorie. *I. Leipzig*, 1927.

[22] Bruce C Berndt. *Ramanujan's notebooks.* Springer Science & Business Media, 2012.

[23] William B Jones and WJ Thron. Survey of continued fraction methods of solving moment problems and related topics. In *Analytic theory of continued fractions*, pages 4–37. Springer, 1982.

[24] Jonathan M Borwein, Peter B Borwein, and David H Bailey. Ramanujan, modular equations, and approximations to pi or how to compute one billion digits of pi. *The American Mathematical Monthly*, 96(3):201–219, 1989.

[25] Roger Apéry. Irrationalité de ζ (2) et ζ (3). *Astérisque*, 61(11-13):1, 1979.

A Additional Results by the MITM-RF Algorithm

In this section we show a sample of generalized continued fractions (GCFs) that were all found by our MITM-RF algorithm, and are listed here in addition to the ones presented in the main text. Our MITM-RFs algorithm was able to reproduce previously known and proven results, along with new RFs which we present here as conjectures.

Convergence	Known / New	Formula	Polynomials
Super-exponential	new	$\frac{1}{e-1} = 1 + \frac{1}{1+\frac{-1}{1+\frac{3}{1+\dots}}}$	$a_{n_1} = 1,\ b_{n_1} = -1$ $a_{n_2} = 1,\ b_{n_2} = n$
	known	$e - 1 = 1 + \frac{2}{2+\frac{3}{3+\frac{4}{4+\dots}}}$	$a_n = 1 + n, b_n = 1 + n$
Exponential	new	$\frac{4}{\pi} = 1 + \frac{1\cdot(3-2\cdot1)}{1+3\cdot1+\frac{2\cdot(3-2\cdot2)}{1+3\cdot2+\frac{3\cdot(3-2\cdot3)}{1+3\cdot3+\dots}}}$	$a_n = 1 + 3n,\ b_n = n(3 - 2n)$
	new	$\frac{2}{\pi} = 1 + \frac{1\cdot(1-2\cdot1)}{1+3\cdot1+\frac{2\cdot(1-2\cdot2)}{1+3\cdot2+\frac{3\cdot(1-2\cdot3)}{1+3\cdot3+\dots}}}$	$a_n = 1 + 3n,\ b_n = n(1 - 2n)$
	new	$\frac{2}{\pi+4} = 1 + \frac{1\cdot(3-2\cdot1)}{1-3\cdot1+\frac{2\cdot(3-2\cdot2)}{1-3\cdot2+\frac{3\cdot(3-2\cdot3)}{1-3\cdot3+\dots}}}$	$a_n = 1 - 3n,\ b_n = n(3 - 2n)$
	new	$\frac{2}{\pi+2} = 2 + \frac{1\cdot(1-2\cdot1)}{2+3+\frac{2\cdot(1-2\cdot2)}{2+3\cdot2+\frac{3\cdot(1-2\cdot3)}{2+3\cdot3+\dots}}}$	$a_n = 2 + 3n,\ b_n = n(1 - 2n)$
	known	$\frac{4}{\pi} = 1 + \frac{1^2}{3+\frac{2^2}{5+\frac{3^2}{7+\dots}}}$	$a_n = 1 + 2n,\ b_n = n^2$
Polynomial	known	$\frac{4}{\pi} = 1 + \frac{1^2}{2+\frac{3^2}{2+\frac{5^2}{2+\dots}}}$	$a_n = 2,\ b_n = (2n - 1)^2$
	known	$\pi + 3 = 6 + \frac{1^2}{6+\frac{3^2}{6+\frac{5^2}{6+\dots}}}$	$a_n = 6,\ b_n = (2n - 1)^2$
	known	$\frac{2}{2-\pi} = 3 - \frac{2\cdot3}{1-\frac{1\cdot2}{3-\frac{4\cdot5}{1-\dots}}}$	$a_{n_1} = 3,\ b_{n_1} = 2n(2n + 1)$ $a_{n_2} = 1,\ b_{n_2} = n(n - 1)$
	known	$\frac{6}{\pi^2-6} = 1 + \frac{1^2}{1+\frac{1\cdot2}{1+\frac{2^2}{1+\dots}}}$	$a_{n_1} = 1,\ b_{n_1} = \frac{(n+1)^2}{2}$ $a_{n_2} = 1,\ b_{n_2} = \frac{n\left(\frac{n}{2}+1\right)}{2}$

Table 3: (Conjectures 5-9) Sample of automatically generated conjectures for mathematical formulas of fundamental constants, as generated by our proposed Ramanujan Machine by applying the Meet-In-The-Middle Regular Formula (MITM-RF) algorithm. We mark which results are previously unknown conjectures to the best of our knowledge. For each RF we provide the convergence rates and polynomials.

B Structured MITM-RF

From previously known representations for Apéry's constant as infinite sums, and by deriving GCFs from infinite sums using Euler's continued fraction identity, we found formulas for Apéry's constant, and noted that they are commonly constructed with high degree polynomials as partial numerators and partial denominators (with the b_n polynomial having double the degree of the a_n polynomial). Yet, they can be transformed into sparse polynomials. Then, by enumerating only on sparse integer polynomials in the MITM algorithm, we were able to find the RFs for Apéry's constant (Table 4).

Formula	Polynomials
$\frac{1}{\zeta(3)} = 0^3 + 1^3 - \frac{1^6}{1^3+2^3-\frac{2^2}{2^3+3^3-\frac{3^6}{3^3+4^3-\ldots}}}$	$a_n = n^3 + (n+1)^3$ $b_n = -n^6$
$\frac{5}{2\zeta(3)} = 2 + \frac{2\cdot1^5\cdot1}{2+1\cdot3\cdot7+\frac{2\cdot2^5\cdot3}{2+1\cdot4\cdot10+\frac{2\cdot3^5\cdot5}{2+1\cdot5\cdot13+\ldots}}}$	$a_n = 2 + n(2+n)(4+3n)$ $b_n = 2n^5(2n-1)$

Table 4: Two Apéry GCF and their polynomials, matching formulas 1 & 2 in the main text. The top converges with polynomial rate, and the bottom converges with exponential rate.

C GCF Convergence Rate

The method detailed in Section 3 requires estimating the expected accuracy from finite approximation of GCFs. In this section we characterize the convergence rate of the GCFs, as well as a trick that improves this convergence rate for the exponential case.

For two sets of numbers $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ we define the generalized continued fraction (GCF) generated by them as

$$[a_0; (b_1, a_1), (b_2, a_2), \ldots] \quad := \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \ldots}}$$

and the partial GCF as

$$\eta_n \quad := \quad [a_0; (b_1, a_1), (b_2, a_2), \ldots, (b_n, a_n)]$$

if the limit exists, we define:

$$\eta \quad := \quad \lim_{n\to\infty} \eta_n$$

We also define the tail:

$$\tau_n \quad := \quad [a_n; (b_{n+1}, a_{n+1}), (b_{n+2}, a_{n+2}), \ldots]$$

From there it follows that:

$$\eta \quad = \quad \frac{p(\tau_n)}{q(\tau_n)}$$

where p, q are polynomials of degree $n-1$ whose coefficients depend on $\{a_i\}_{i=0}^{n-1}, \{b_i\}_{i=1}^{n}$. Specifically, for $a_i, b_i \in \mathbb{Z}$ we have $p, q \in \mathbb{Z}[x]$.

It was shown (Jones & Thron, 1982) that the partial GCF η_n can be computed as a series of Matrix-Vector multiplications:

$$\begin{aligned}
\begin{pmatrix} p_0 \\ p_{-1} \end{pmatrix} \quad &:= \quad \begin{pmatrix} a_0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} q_0 \\ q_{-1} \end{pmatrix} \quad &:= \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} p_{n+1} & q_{n+1} \\ p_n & q_n \end{pmatrix} \quad &:= \quad \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} \\
&\Downarrow \\
p_{n+1} \quad &= \quad a_n p_n + b_n p_{n-1} \\
q_{n+1} \quad &= \quad a_n q_n + b_n q_{n-1}
\end{aligned}$$

From which η_n can be calculated like so:

$$\eta_n = \frac{p_n}{q_n}$$

In the following sections we discuss GCFs with integer polynomials for a_n and b_n:

$$\begin{aligned} a(x), b(x) &\in \mathbb{Z}[x] \\ a_n &= a(n) \\ b_n &= b(n) \end{aligned}$$

which we will abbreviate as GCFPs.

C.1 GCFP Error Bound

Taking the determinant of the above linear equation, we can deduce the following expression for the matrix determinant:

$$\begin{aligned} p_{n+1}q_n - q_{n+1}p_n &= (-1)^n \prod_{i=1}^{n} b_i \\ &\Downarrow \\ \eta_{n+1} - \eta_n &= (-1)^n \frac{\prod_{i=1}^{n} b_i}{q_{n+1}q_n} \end{aligned}$$

If b_i has constant sign for all $i > k$ from some k, we get a the following Leibniz series:

$$\sum_{i=k}^{\infty} \eta_{n+1} - \eta_n$$

Since:

$$\eta_{n+1} = \eta_k + \sum_{i=k}^{\infty} \eta_{i+1} - \eta_i = \eta_k + \sum_{i=k}^{\infty} (-1)^n \frac{\prod_{i=1}^{n} b_i}{q_{n+1}q_n}$$

the following relation is achieved:

$$\forall n \geq k \quad \eta_{2n+\kappa} \quad \leq \lim_{n\to\infty} \eta_{2n+\kappa} \leq \lim_{n\to\infty} \eta_{2n+\kappa-1} \quad \leq \eta_{2n+\kappa-1}$$

where $\kappa \in \{0,1\}$, depending on the sign of $\prod_{i=1}^{k} b_i$ and $k \mod 2$. Hence we get that $\exists \eta$ and

$$|\eta - \eta_n| \leq \left| \frac{\prod_{i=1}^{n} b_i}{q_{n+1}q_n} \right|$$

C.2 1-Periodic GCF

A GCFP is called k-periodic if $\forall n \in \mathbb{N}\ a_n = a_{n+k},\ b_n = b_{n+k}$. A 1-periodic GCFP is one of the form:

$$a + \frac{b}{a + \frac{b}{a+\ldots}}$$

hence for $(w_n) = (p_n)$ or $(w_n) = (q_n)$:

$$\begin{pmatrix} w_{n+1} \\ w_n \end{pmatrix} := \underbrace{\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}}_{\substack{\text{promoter} \\ \text{matrix}}} \begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix}$$

For $a^2 > -4b$, we find that the promoter matrix is real-diagonalizable

$$\begin{aligned}
\text{eigvals}\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} &= \left\{\frac{a \pm \sqrt{a^2+4b}}{2}\right\} = \{\lambda_\pm\} \\
\text{eigvecs}\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} &= \left\{\begin{pmatrix} \lambda_\pm \\ 1 \end{pmatrix}\right\} = \{\mathbf{v}_\pm\} \\
&\Downarrow \\
\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} &= \kappa_+\mathbf{v}_+ + \kappa_-\mathbf{v}_- \\
\begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} &= \lambda_+^n\kappa_+\mathbf{v}_+ + \lambda_-^n\kappa_-\mathbf{v}_-
\end{aligned}$$

And from there, the decomposition for p, q is:

$$\begin{aligned}
\begin{pmatrix} p_0 \\ p_{-1} \end{pmatrix} &= \begin{pmatrix} a_0 \\ 1 \end{pmatrix} = \frac{\lambda_+}{\sqrt{a^2+4b}}\mathbf{v}_+ - \frac{\lambda_-}{\sqrt{a^2+4b}}\mathbf{v}_- \\
\begin{pmatrix} q_0 \\ q_{-1} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\mathbf{v}_+ - \mathbf{v}_-}{\sqrt{a^2-4b}}
\end{aligned}$$

Thus

$$\frac{p_{n-1}}{q_{n-1}} = \frac{\frac{1}{\sqrt{a^2+4b}}\left(\lambda_+^{n+1} - \lambda_-^{n+1}\right)}{\frac{1}{\sqrt{a^2+4b}}\left(\lambda_+^{n} - \lambda_-^{n}\right)} = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+^{n} - \lambda_-^{n}}$$

For $a > 0$ we get that $|\lambda_+| > |\lambda_-| \geq 0$, hence:

$$\begin{aligned}
\frac{p_{n-1}}{q_{n-1}} &= \lambda_+ \frac{1 - \left(\frac{\lambda_-}{\lambda_+}\right)^{n+1}}{1 - \left(\frac{\lambda_-}{\lambda_+}\right)^{n}} \\
\lim_{n\to\infty} \eta_n &= \lambda_+
\end{aligned}$$

While in the case $a < 0$ we have $|\lambda_-| > |\lambda_+| \geq 0$, which in turn results in:

$$\lim_{n\to\infty} \eta_n = -\lambda_-$$

Note that if we knew $\exists \lim_{n\to\infty} \eta_n$ then: $\eta = a + \frac{b}{\eta}$, yielding a quadratic equation with the same results.

C.3 Types Of Convergence

Not every continued fraction converges. In the case it does, its rate of convergence is either: exponential, super-exponential, or sub-exponential (which seems to be at a polynomial rate, however it is yet to be proven). When the continued fraction does not converge, it may oscillate between a set of values or "converge" to a discrete oscillating cycle, meaning that for a k-oscillation with values $\{o_i\}_{i=0}^{k-1}$, we have $\lim_{n\to\infty} |\eta_n - o_{n \bmod k}| = 0$.

In the following parts, we analyze the GCFP behaviour with regard to its defining polynomials a, b. We'll use the following notation:

$$\begin{aligned}
d_a &:= \deg(a) \\
d_b &:= \deg(b) \\
a(x) &= \sum_{j=1}^{d_a} \alpha_j x^j \\
b(x) &= \sum_{i=0}^{d_b} \beta_j x^i
\end{aligned}$$

For an easier analysis of the GCFP behavior, we use the equivalence transformation and define its semi-canonical form as[3]:

$$\begin{aligned}
\forall n \in \mathbb{N}\ c_n &:= \frac{b_n}{a_{n-1}a_n} \\
a_0\left(1+\frac{\frac{b_1}{a_0 a_1}}{1+\frac{\frac{b_2}{a_1 a_2}}{1+\ldots}}\right) &=: [a_0;(c_1,c_2,\ldots)]
\end{aligned}$$

From there it follows:

$$\eta_n = [a_0;(c_1,\ldots,c_n)]$$

Unless stated otherwise we'll regard only the main part of the above GCFP:

$$1+\frac{\frac{b_1}{a_0 a_1}}{1+\frac{\frac{b_2}{a_1 a_2}}{1+\ldots}} = 1+\frac{c_1}{1+\frac{c_2}{1+\ldots}}$$

We now recognize 3 distinct cases. In the first case, denoted as the exponential case we have:

$$\begin{aligned}
d_b &= 2d_a \\
&\Downarrow \\
\lim_{n\to\infty} c_n &= \frac{\beta_{d_b}}{\alpha_{d_a}^2}
\end{aligned}$$

The second case, denoted as the super-exponential case we have:

$$\begin{aligned}
d_b &< 2d_a \\
&\Downarrow \\
\lim_{n\to\infty} c_n &= 0
\end{aligned}$$

And finally, the third case, denoted as the sub-exponential or the polynomial case we have:

$$\begin{aligned}
d_b &> 2d_a \\
&\Downarrow \\
\lim_{n\to\infty} c_n &= \operatorname{sign}(\beta_{d_b})\cdot\infty
\end{aligned}$$

For all cases, from some point, $c_n \approx \frac{\beta_{d_b}}{\alpha_{d_a}^2} n^{d_b-2d_a}$, meaning that $\operatorname{sign}(c_n) = \operatorname{sign}(\beta_{d_b})$. Thus $|q_n| = |q_{n-1} + c_n q_{n-2}|$ is monotonically increasing.
Based on the observation that η is a rational function of τ_n, it's enough to show that the above claims for the convergence rate apply for the tail τ_n for some n.

[3] This is well defined, as we're examining the tail's behavior. Therefore neglect n's for which $a_n = 0$, as they are finite.

C.3.1 Exponential

From some point $c_n \approx \frac{\beta_{d_b}}{\alpha_{d_a}^2} =: c$, therfore q_n increases as a generalized Fibonacci series. Specifically, it does not change sign. Therefore we can now refer to the 1-periodic case, as equivalent results can be derived here similarly to Section C.2, since:

$$\begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & c \\ 1 & 0 \end{pmatrix}^{n-k} \begin{pmatrix} q_k \\ q_{k-1} \end{pmatrix}$$

So with the same condition on the determinant, which here translate to:

$$\begin{aligned} 4c+1 &> 0 \\ &\Updownarrow \\ 4\beta_{d_b} &> -\alpha_{d_a}^2 \end{aligned}$$

We get that the GCFP converges to λ_+ (WLOG, we assume that $|\lambda_+| > |\lambda_-|$. The $-\lambda_-$ case is similar), and therefore:

$$\eta_n \approx \kappa\lambda_+ \frac{1-\left(\frac{\lambda_-}{\lambda_+}\right)^{n+1}}{1-\left(\frac{\lambda_-}{\lambda_+}\right)^{n}}$$

where κ is a constant arising from the point k at which we assume $c_i \approx c$. We then receive:

$$\begin{aligned} |\eta-\eta_n| &\approx \kappa\lambda_+ \left|1-\frac{1-\left(\frac{\lambda_-}{\lambda_+}\right)^{n+1}}{1-\left(\frac{\lambda_-}{\lambda_+}\right)^{n}}\right| \\ &= \kappa\lambda_+ \left|\frac{\lambda_-}{\lambda_+}\right|^n \left|\frac{-1+\left(\frac{\lambda_-}{\lambda_+}\right)}{1-\left(\frac{\lambda_-}{\lambda_+}\right)^{n}}\right| \\ &\leq \kappa\lambda_+ \left|-1+\left(\frac{\lambda_-}{\lambda_+}\right)\right| \left|\frac{\lambda_-}{\lambda_+}\right|^n \end{aligned}$$

Since $|\lambda_-| < |\lambda_+|$ we get an exponentially decreasing value, hence:

$$\begin{aligned} |\eta-\eta_n| &\leq \frac{\prod_{i=1}^n c_i}{|q_{n+1}q_n|} \leq \kappa_3 \left|\frac{4c}{4c+2+2\sqrt{4c+1}}\right|^{(n-k)} \\ &\Downarrow \\ |\eta-\eta_n| &\leq \kappa_3 \exp\left(-\kappa_4 n\right) \end{aligned}$$

Notice that this result is dependent on κ, which makes it an upper bound for the error rather then the exact error (who's calculation is equivalent to the calculation of η). With a more careful calculation the exponent parameters can be found to get a tighter bound on the error. In sake of brevity, these calculations are omitted.

C.3.2 Super-exponential

Again we assume WLOG that $\beta_{d_b} > 0$. Threfore for all $n \geq k$ for some k :

$$c_n \approx \frac{\beta_{d_b}}{\alpha_{d_a}^2} n^{\overbrace{d_b - 2d_a}^{<0}}$$

Also, since $\forall n > k\,:\, c_n > 0$. From there we get that $\forall n > k:\; q_n > q_k$. Which in turn results in:

$$\begin{aligned}
|\eta - \eta_n| &\leq \left|\frac{\prod_{i=1}^{n} c_i}{q_{n+1}q_n}\right| \leq \left|\frac{\prod_{i=1}^{n} c_i}{q_k^2}\right| \\
&\approx \kappa_1 \left(\frac{\beta_{d_b}}{\alpha_{2\,d_a}}\right)^{n-k} \left(\frac{n!}{k!}\right)^{d_b-2d_a} \\
&= \kappa_2 \frac{\left(\frac{\beta_{d_b}}{\alpha_{2\,d_a}}\right)^{n-k}}{(n!)^{2d_a-d_b}}
\end{aligned}$$

Since $\frac{exp(n)}{n!}$ is decreasing super-exponentially, the desired result is obtained.

C.3.3 Sub-Exponential

The case satisfying the determinant constraint $4\beta_{d_b} > -\alpha_{d_a}^2$ can be seen as a limit of the exponential convergence case with $c \to \infty$, therefore the derived convergence is sub-exponential. We believe this sub-exponential convergence to be polynomial.

C.3.4 Tail Estimation

In the case of an exponentially converging GCFP, we found that from some point the tail is approximately:

$$1 + \frac{c}{1 + \frac{c}{1+\ldots}}$$

We calculated the convergence value of 1-periodic GCFs like this earlier. Therefore, we can improve a GCFP calculation by substituting this tail at the final step. The accuracy improvement wasn't analyzed, but empiric results display an improvement of a fixed number of digits (for any large n). This in turn allowed us to improve the complexity of the MITM-RF algorithm.

D Collaborative Algorithm-Enhanced Mathematics

The Ramanujan Machine in its most general sense can be seen as a methodology to generate conjectures on fundamental constants. The more computational power and the more time the algorithm runs on a selected space of parameters, the more conjectures it may generate. Moreover, since the Ramanujan Machine produces conjectures on fundamental constants but not their proofs, we realize that computational power as well as proving power (i.e. time spent by an intelligent being trying to prove or refute a conjecture) are key assets for making the Ramanujan Machine more prolific. It is the goal of this section to discuss how one may leverage these facts about the Ramanujan Machine methodology to inspire the wider community about mathematics and number theory.

We created the Ramanujan Machine as an open source project that is fully available to the community on `www.RamanujanMachine.com`. Soon, with our ongoing development, individuals around the world would be able to donate their computational power to the mission of discovering new mathematical structures and mathematical equations by downloading the Ramanujan Machine screen saver on the website. Similarly to SETI (Search for Extraterrestrial Intelligence), we plan to have the Ramanujan Machine screen saver distribute via BOINC the various computational tasks to every computer in the network, so when a computer is idle, the Ramanujan Machine is initiated.

We believe this methodology can inspire the greater community about mathematics. In order to achieve this goal, the site `www.RamanujanMachine.com` includes an up-to-date record of some (and in a short time every) conjecture generated by the machine. When a specific computer in the network discovers a new conjecture, the owner of the laptop will receive the credit for contributing his or her computer power to discover the conjecture and the credit is maintained in a leadership board. Since the Ramanujan Machine is a conjecture-generating machine (similarly to much of the work of Ramanujan himself), we let the community suggest proofs for each conjecture, thus honorably claiming affiliation to Ramanujan's legacy, and introducing an algorithm-enhanced approach for collaborative research.

It is important to emphasize that the methodology introduced in this work can be expanded far beyond continued fractions, number theory or mathematics. The Ramanujan Machine is an example of a broader methodology that has three core elements in its pipeline, as shown in the main text (Fig. 1).

E Further Information about the Descent&Repel Method and Results

This section provides an additional example of the Descent&Repel optimization process (in Fig. 5), in addition to providing Table 5 with further information about the process presented in the main text (in Fig. 4). The parameters chosen for Fig. 4 illustrate the optimization steps relatively clearly, however without converging to any real solution. In Fig. 5, we present a similar illustration (Fig. 5), presenting the convergence to $e = 3 + \frac{-1}{4+\frac{-2}{5+\frac{-3}{6+\dots}}}$.

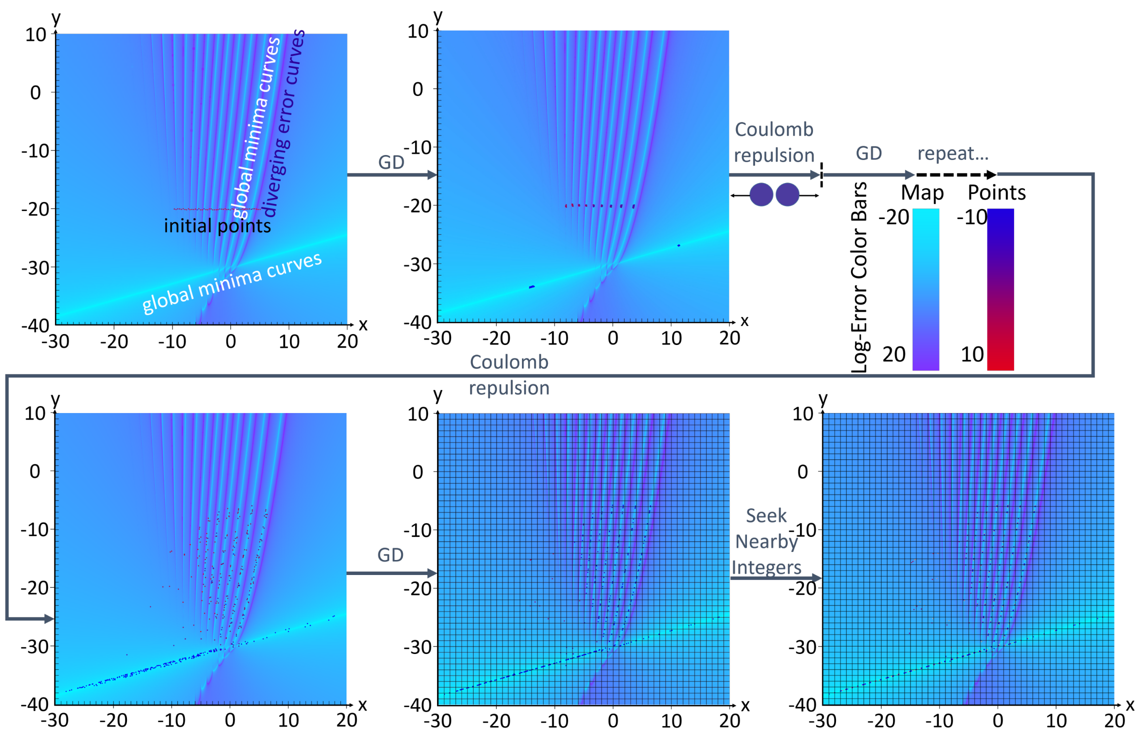

Figure 5: Descent&Repel illustration, as in Fig. 4. Here the showcased scenario is of that of the restoration of our previous result (new, found by the MITM-GCF algorithm) $e = 3 + \frac{-1}{4+\frac{-2}{5+\frac{-3}{6+\dots}}}$. The converging point is the one at $(x, y) = (4, -1)$.

Below are the parameters required to reproduce the results in Fig. 4 and Fig. 5.

Fig. Number	Parameters	Values
Fig. 4	$a(n)$	n
	$b(n)$	$n^2 + ny + x$
	x range	$[-20, 20]$
	y range	$[-20, 20]$
	fraction depth	10
	constant	π
	initial points	600, uniform at $y \in [-15, 15]$ and $x = 10$
Fig. 5	$a(n)$	$n + x$
	$b(n)$	$-n + y$
	x range	$[-30, 20]$
	y range	$[-40, 10]$
	fraction depth	20
	constant	$e - 3$
	initial points	500, uniform at $y = -20$ and $x \in [-10, 5]$

Table 5: Execution settings required to reproduce the Descent&Repel maps (Fig. 4, Fig. 5). Here, a, b are similar to the polynomials α, β which define the GCF, but the RF is of the form: $\frac{b_0}{a_0 + \frac{b_1}{a_1 + \cdots}}$.