An infinite family of results for Polylogarithm function. July, 2022

In July 2022, users from our BOINC Community started discovering results that converged to values of the fourth order of the Polylogarithm function.

We've found that all of those results' polynomials matched this structures:

$$a_n = c_1 n^4 + c_2(n+1)^4$$
$$b_n = -c_1 c_2 n^8$$

Notice that this notation is equivalent to setting $c_1 = 1$ and choosing c_2 to be rational:

$$a_n = n^4 + c(n+1)^4$$

$$b_n = -cn^8$$

We've found that for every $c \geq 1$ the continued fraction will converge to

$$Li_4(1/c)$$

And when $c < 1$ it will converge to zero. By changing the polynomial powers, we've manage to create a general scheme for every order of the Polylogarithm function. By using the following polynomials -

$$PCF\begin{pmatrix} a_n = n^s + c(n+1)^s \\ b_n = -cn^{2s} \end{pmatrix} = Li_s(1/c)$$

We proved this family by using Euler's continued fraction formula, and the series that defines polylog -

$$Li_s(z) = \sum \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \frac{z^4}{4^s} = z + z\frac{z \cdot 1^s}{2^s} + z\frac{z \cdot 1^s}{2^s}\frac{z \cdot 2^s}{3^s} + z\frac{z \cdot 1^s}{2^s}\frac{z \cdot 2^s}{3^s}\frac{z \cdot 3^s}{4^s}$$

We denote $r_n = \frac{z(n-1)^s}{n^s}$ and get from Euler's continued fraction formula -

$$Li_s(z) = \cfrac{1}{1 - \cfrac{r_1}{1+r_1 - \cfrac{\cdots}{\cdots - \cfrac{r_{n-1}}{1+r_{n-1} - \cfrac{a_n}{1+a_n - \cdots}}}}}$$

By focusing on the part of the continued fraction that describe the general step -

$$\frac{\cdots}{\cdots - \cfrac{r_n}{1+r_n - \cfrac{r_{n+1}}{1+r_{n+1} - \cdots}}} = \frac{\cdots}{\cdots - \cfrac{\frac{z(n-1)^s}{n^s}}{1+\frac{z(n-1)^s}{n^s} - \cfrac{\frac{zn^s}{(n+1)^s}}{1+\frac{zn^s}{(n+1)^s} - \cdots}}} = \frac{\cdots}{\cdots - \cfrac{z(n-1)^s}{n^s + z(n-1)^s - \cfrac{zn^s}{(n+1)^s + zn^s - \cdots}}}$$

By setting $z = 1/c$ and multiplying the every step in the continued fraction with c/c we get the general scheme suggested here.