RAMANUJAN CHALLENGE PROOFS

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ABSTRACT

To help evaluate the mathematical skills of current AI systems, we present a set of formulas and their solutions for fundamental mathematical constants. These problems are attractive for AI evaluation because they are concrete and can be checked numerically to arbitrary precision, yet proving them may require non-obvious mathematics. Mathematical constants such as π, e, Catalan's constant, and special values of the Riemann zeta function have fascinated mathematicians for centuries. The search for formulas evaluating mathematical constants has produced some of the most beautiful mathematics in the field, especially in cases that yield irrationality proofs or fast convergence rates. Ramanujan's legacy is emblematic of this tradition. The list we provide contains two types of problems: formulas whose proofs are known to the authors but will remain encrypted for a short initial period; and formulas that are not yet proven. We are curious to see the achievements of AI in both cases.

Problem	Name	Contributor
1	Polynomial continued fraction for π	Michael Shalyt[1]
2	Euler's constant γ as an Apéry limit	Rotem Kalisch[1]
3	The sum $\pi + e$ as an Apéry limit	
4	A series of harmonic numbers converging to a polylogarithm combined with zeta values	Carsten Schneider[2]
5	Efficient rational approximation of Catalan's constant G	Hila Barkan[1]
6	A series for $\zeta(2) + \zeta(3)$	
7	Efficient four-term recurrence for $\zeta(2) + \zeta(3)$	Elyasheev Leibtag[3]
8	Very fast rational approximation of $\sqrt{10005}/\pi$	

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1 INTRODUCTION

Recent progress in AI for mathematics has made quantitative evaluation increasingly urgent. We increasingly need university-level and research-level questions that test whether AI systems can contribute to genuine mathematical work. Several recent benchmarks address this need from different directions. RealMath and LemmaBench study mathematical questions drawn from research papers and mathematical forums [1, 2]. A central difficulty is contamination: if a problem or its solution appears in the AI training data, success may reflect retrieval rather than reasoning. One response is to perturb existing problems, as in GSM-Symbolic [3] and ASyMOB [4], reducing dependence on uncontaminated original questions. A stronger response is to use authentic research problems that have not yet appeared publicly. FrontierMath [5], Riemann-Bench [6], and part of Humanity's Last Exam [7] emphasize original difficult questions with structured verification, while First Proof [8] focuses on unpublished research problems whose answers were known to experts.

The present manuscript follows the spirit of First Proof, but concentrates on a more focused domain with a long mathematical tradition: explicit formulas involving fundamental mathematical constants, presented as recurrences, continued fractions, series, and integrals. The most interesting formulas often point to hidden structures (algebraic, group-theoretic, etc.). Historically, such formulas have been signatures of mathematical inspiration, from Ramanujan's "dreams" to Apéry's "garden". This document collects rigorous proofs for each of the problems.

This domain is well suited to quantitative evaluation of AI abilities. On the one hand, candidate formulas for constants can often be tested to thousands of digits, so numerical validation is immediate and objective. On the other hand, converting such evidence into proof may require a wide variety of mathematical tools from various domains. It is often hard to know in advance which approach will be best suited for each formula.

The historical examples are iconic. Ramanujan's formulas for π remain a model of unexpected structure in special values [9]. Apéry's proof of the irrationality of $\zeta(3)$, together with the reinterpretation by Beukers via multiple integrals, showed how recurrence relations and arithmetic estimates can turn an experimental pattern into a theorem [10, 11]. Subsequent work and continued ongoing efforts extend this perspective to odd zeta values, Catalan's constant, Euler's constant, and a wide range of Apéry limits and related special values [12, 13, 14, 15, 16, 17, 18, 19].

Many of the questions collected here lie in or near the modern Apéry-style tradition. Some ask for polynomial continued fractions or holonomic recurrences converging to constants such as π, $\log 2$, γ, $\pi + \mathrm{e}$, or combinations of zeta values. Others concern efficient recurrences with a conjectural exponential rate of convergence, or integrals whose closed forms appear to encode deeper arithmetic structure. At a broader conceptual level, these identities also belong to the world of periods and special values, where numerical discovery frequently precedes structural explanation [20].

These formulas were discovered as *trajectories* in Conservative Matrix Fields (CMFs) generated from hypergeometric and Meijer G-functions, and were produced and manipulated with the `ramanujantools` package [21, 18, 19, 22].

Why are the formulas so complex? Mostly because AI got so good, and can immediately prove most simple forms. Nevertheless, some of the most interesting formulas presented below are complex because no simpler forms are known, as in the example of the recurrence that produces $\pi + e$. We will see whether/when AI can provide new insight into these classic questions.

We present our team's original solutions in this paper, and shall write about the successes of the challenge separately.

2 Questions

Question 1 (Polynomial continued fraction for π)**.** *Let:*

$$a_n = -220n^3 - 484n^2 - 301n - 42, \qquad b_n = 4n^2(2n+1)^2(5n-4)(5n+6). \tag{1}$$

Prove:

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}} = \frac{6}{3-\pi}. \tag{2}$$

Question 2 (Euler's constant γ as an Apéry limit). *For $n \geq 0$, define:*

$$\begin{aligned}0 &= \left(-8n^3-51n^2-105n-68\right)u_n \\ &+ \left(24n^5+337n^4+1833n^3+4818n^2+6092n+2928\right)u_{n-1} \\ &- (n+2)(n+3)\left(24n^5+273n^4+1150n^3+2154n^2+1635n+268\right)u_{n-2} \\ &+ (n+1)(n+2)^4(n+3)\left(8n^3+75n^2+231n+232\right)u_{n-3}\end{aligned}$$

Let p_n, q_n be two solutions of the recurrence, defined by the initial values:

$$p_{-3}=0, \quad p_{-2}=7, \quad p_{-1}=179$$
$$q_{-3}=1, \quad q_{-2}=12, \quad q_{-1}=306$$

Prove

$$\lim_{n\to\infty}\frac{p_n}{q_n}=\gamma,$$

where γ is Euler's constant.

Question 3 (The sum $\pi + e$ as an Apéry limit). *For $n \geq 1$, define*

$$\begin{aligned}0 &= \left(-n^3+2n^2+7n+3\right)u_n \\ &+ (n+2)\left(2n^4+n^3-26n^2-48n-19\right)u_{n-1} \\ &+ (n+2)\left(n^6+9n^5+8n^4-87n^3-249n^2-234n-68\right)u_{n-2} \\ &+ (n+1)^2(n+2)\left(2n^5+3n^4-13n^3-21n^2+4\right)u_{n-3} \\ &- n^3(n+1)^2(n+2)\left(n^3+n^2-8n-11\right)u_{n-4}.\end{aligned}$$

Let p_n, q_n be two solutions of the recurrence, defined by the initial values:

$$p_{-3}=1, \quad p_{-2}=1, \quad p_{-1}=20, \quad p_0=296$$
$$q_{-3}=1, \quad q_{-2}=0, \quad q_{-1}=4, \quad q_0=48$$

Prove

$$\lim_{n\to\infty}\frac{p_n}{q_n}=\pi+e.$$

Question 4 (Harmonic numbers, a polylogarithm and zeta values). *Prove:*

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\binom{m}{k}^2 H_k^2}{(m+1)^2\binom{2m}{m}} =$$

$$20\,\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{5}{6}\log^4(2)+10\zeta(2)-\frac{65}{9}\zeta(2)^2-\log^2(2)(12+5\zeta(2))+\frac{1}{2}\zeta(3)+\log(2)\left(\frac{35}{2}\zeta(3)-16\right),$$

where $H_0 = 0$ and for $k \geq 1$, H_k is the k-th harmonic number.

Question 5 (Efficient rational approximation of Catalan's constant G). *For $n \geq 0$, let*

$$M(n) = \left(m_{ij}(n)\right)_{1\leq i,j\leq 3},$$

where

$$\begin{aligned}m_{11}(n) &= (-2n-5)(n+3)^2(136n^4+1424n^3+5548n^2+9551n+6141), \\ m_{12}(n) &= 384n^6+6384n^5+44168n^4+162698n^3+336377n^2+369933n+169011, \\ m_{13}(n) &= -480n^4-4980n^3-19210n^2-32690n-20730, \\ m_{21}(n) &= (n+2)^2(n+3)^2(4n+10)(48n^3+386n^2+1017n+879), \\ m_{22}(n) &= (n+2)^2(-272n^5-3848n^4-21732n^3-61184n^2-85761n-47808), \\ m_{23}(n) &= (n+2)^2(320n^3+2540n^2+6610n+5640), \\ m_{31}(n) &= (-4n-10)(n+2)^2(n+3)^2(32n^4+302n^3+1037n^2+1530n+813), \\ m_{32}(n) &= (n+2)^2(192n^6+2984n^5+19116n^4+64452n^3+120256n^2+117279n+46476), \\ m_{33}(n) &= (n+2)^2(-16n^5-408n^4-2912n^3-8884n^2-12254n-6240).\end{aligned}$$

For $N \geq 0$, define

$$\mathcal{M}_N = M(0)M(1)\cdots M(N-1).$$

Choose initial conditions

$$A = \begin{pmatrix} 30921 & -32972 & 8240 \\ 33750 & -36000 & 9000 \end{pmatrix}.$$

Write

$$A\mathcal{M}_N = \begin{pmatrix} P_{N,1} & P_{N,2} & P_{N,3} \\ Q_{N,1} & Q_{N,2} & Q_{N,3} \end{pmatrix}.$$

Then, for each $j = 1, 2, 3$, prove:

$$\lim_{N\to\infty} \frac{P_{N,j}}{Q_{N,j}} = G,$$

where

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

is Catalan's constant.

Question 6 (A series for $\zeta(2) + \zeta(3)$)**.** *Let $(u_n)_{n\geq 1}$ be the sequence defined by*

$$u_1 = -\frac{93}{4480}, \qquad u_2 = -\frac{117}{14000},$$

and, for $n \geq 3$, by the recurrence

$$\begin{aligned} 0 = &- 2(n+3)^3(2n+5)(3n+5)u_n \\ &+ (n+2)^2(15n^3 + 85n^2 + 155n + 93)u_{n-1} \\ &- (n+1)^3(n+2)(3n+8)u_{n-2}. \end{aligned}$$

Prove:

$$\frac{2077}{720} + \sum_{j=1}^{\infty} u_j = \zeta(2) + \zeta(3).$$

Question 7 (Efficient four-term recurrence for $\zeta(2) + \zeta(3)$)**.** *For $n \geq 0$, define*

$A_n = 1024(2n+5)^4(2n+7)^3(2n+9)^3\big(946n^2 + 6407n + 10860\big)$,
$B_n = 128(2n+7)^3(2n+9)^3\big(104060n^6 + 1745370n^5 + 12145238n^4 + 44886481n^3 + 92943995n^2 + 102256019n + 46709052\big)$,
$C_n = 16(n+3)^4(2n+9)^3\big(3784n^5 + 57792n^4 + 351019n^3 + 1059230n^2 + 1587211n + 944620\big)$,
$D_n = (n+3)^4(n+4)^6\big(946n^2 + 4515n + 5399\big)$.

Let $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ be the two solutions of the recurrence

$$u_{n+1} = \frac{B_n}{A_n}\,u_n - \frac{C_{n-1}}{A_{n-1}}\,u_{n-1} + \frac{D_{n-2}}{A_{n-2}}\,u_{n-2}, \qquad n \geq 2,$$

with initial conditions

$$p_0 = -612218384750, \qquad p_1 = -\frac{9525021973931919}{18100}, \qquad p_2 = -\frac{29561828382772029}{65380},$$

and

$$q_0 = -215040420000, \qquad q_1 = -\frac{167282265043404}{905}, \qquad q_2 = -\frac{964185327658080}{6071}.$$

Prove:

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \zeta(2) + \zeta(3).$$

Question 8 (Very fast rational approximation of $\sqrt{10005}/\pi$)**.** *Let*

$$R = 151931373056001, \qquad u = 2n+3, \qquad w = u(3u-2)(3u+2).$$

For $n \geq 0$, let

$$M(n) = \begin{pmatrix} \dfrac{a_1}{w} & \dfrac{a_2}{w} & \dfrac{a_3}{w} & \dfrac{144R(u-1)^2}{w} \\ -u^3 & -3u^2 & -3u & -1 \\ \dfrac{b_1}{144R} & -\dfrac{b_2}{72R} & -\dfrac{b_3}{36R} & -\dfrac{2u+2R-7}{2R} \\ \dfrac{c_1}{288R^2} & \dfrac{c_2}{144R^2} & \dfrac{c_3}{72R^2} & \dfrac{c_4}{4R^2} \end{pmatrix},$$

where

$$\begin{aligned} a_1 &= (144R-99)u^5 - (288R-333)u^4 + (144R-229)u^3 - 114u^2 + 40u + 64, \\ a_2 &= (432R-243)u^4 - (864R-909)u^3 + (432R-868)u^2 - 80u + 272, \\ a_3 &= (432R-153)u^3 - (864R-648)u^2 + (432R-860)u + 360, \end{aligned}$$

$$\begin{aligned} b_1 &= 9u^4 - (144R-63)u^3 + 158u^2 + 168u + 64, \\ b_2 &= 36u^3 + (216R-189)u^2 - 316u - 168, \\ b_3 &= 54u^2 + (108R-189)u - 158, \end{aligned}$$

and

$$\begin{aligned} c_1 &= 18u^5 + (54R+45)u^4 - (288R^2 - 378R + 251)u^3 \\ &\qquad + (948R-1086)u^2 + (1008R-1384)u + (384R-576), \\ c_2 &= (153R-72)u^4 - (657R-702)u^3 - (432R^2 - 1292R + 1069)u^2 \\ &\qquad + (2064R-2508)u + (1072R-1512), \\ c_3 &= (180R-108)u^3 - (891R-864)u^2 - (216R^2 - 1450R + 1385)u \\ &\qquad + (1116R-1422), \\ c_4 &= (6R-4)u^2 - (33R-32)u - (4R^2 - 58R - 236337691420383). \end{aligned}$$

For $N \geq 0$, define

$$\mathcal{M}_N = M(0)M(1)\cdots M(N-1),$$

with the convention that $\mathcal{M}_0 = I$.

Choose initial conditions

$$A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \end{pmatrix},$$

where

$$\begin{aligned} (A_1, A_2, A_3, A_4) = (&3716930576044225276144 1, 111507917281327441564208, \\ &111507917281327599720129, 37169305760442410917362), \end{aligned}$$

and

$$\begin{aligned} (B_1, B_2, B_3, B_4) = (&116741636154263969232 0, 350224908462789613216 0, \\ &350224908462787969728 0, 116741636154262272384 0). \end{aligned}$$

Write

$$A\mathcal{M}_N = \begin{pmatrix} P_{N,1} & P_{N,2} & P_{N,3} & P_{N,4} \\ Q_{N,1} & Q_{N,2} & Q_{N,3} & Q_{N,4} \end{pmatrix}.$$

Then, for each $j = 1, 2, 3, 4$:

$$\lim_{N\to\infty} \frac{P_{N,j}}{Q_{N,j}} = \frac{\sqrt{10005}}{\pi}.$$

3 Preliminaries

We gather here the definitions, theorems, and computational principles that recur across the proofs.

3.1 Conservative Matrix Fields and trajectories

A Conservative Matrix Field (CMF) [18, 19] is a discrete geometric space in which each *trajectory* (an initial point together with an integer direction vector) defines a recurrence relation. The CMFs used below are built from special functions: each carries a row "state-vector" whose coordinates are the function and its Θ-derivatives, and a family of basis matrices that shift the state-vector by one unit in each coordinate direction (these follow from the contiguous relations of the underlying function). A step along a fixed direction v is realised by a *trajectory matrix* $T(n)$ formed as an ordered product of basis matrices.

As an example, the ${}_2F_1$ CMF is generated from the ${}_2F_1$ hypergeometric function at argument $w = 1/2$ [22]. The state-vector is

$$\mathcal{V}(x,y,z) = \left[{}_2F_1\left(x,y;z;\frac{1}{2}\right), \Theta\,{}_2F_1\left(x,y;z;\frac{1}{2}\right)\right], \qquad \Theta = w\frac{d}{dw},$$

and the basis matrices (see Appendix G.2 in [22]) are

$$M_x(x,y,z) = \begin{pmatrix} 1 & y \\ x^{-1} & (2x+y-2z+2)x^{-1} \end{pmatrix}, \tag{3}$$

$$M_y(x,y,z) = \begin{pmatrix} 1 & x \\ y^{-1} & (x+2y-2z+2)y^{-1} \end{pmatrix}, \tag{4}$$

$$M_z(x,y,z) = \begin{pmatrix} \dfrac{-z(x+y-z)}{(x-z)(y-z)} & \dfrac{xyz}{(x-z)(y-z)} \\ \dfrac{z}{(x-z)(y-z)} & \dfrac{-z^2}{(x-z)(y-z)} \end{pmatrix}. \tag{5}$$

They shift the state-vector in the coordinate directions:

$$\begin{aligned} \mathcal{V}(x,y,z)M_x(x,y,z) &= \mathcal{V}(x+1,y,z), \\ \mathcal{V}(x,y,z)M_y(x,y,z) &= \mathcal{V}(x,y+1,z), \\ \mathcal{V}(x,y,z)M_z(x,y,z) &= \mathcal{V}(x,y,z+1). \end{aligned}$$

The same construction, with the Meijer G- or ${}_4F_3$-state vectors of Section 3.8, produces the higher-rank trajectories used in the approximation problems.

3.2 Polynomial Continued Fractions

Definition 1. A *Polynomial Continued Fraction* (PCF) is an expression defined by two polynomials $a_n, b_n \in \mathbb{Z}[n]$:

$$\cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \ddots}}.$$

The expression, evaluated at depth n, satisfies a recurrence of order 2:

$$u_n = a_n u_{n-1} + b_n u_{n-2}$$

3.3 Euler's integral

Lemma 1 (Euler's beta integral)**.** *For reals $c > b > 0$ and $w \notin [1,\infty)$, the hypergeometric function ${}_2F_1(a,b;c;w)$ can be represented as*

$${}_2F_1(a,b;c;w) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-wt)^{-a}\,dt.$$

This is the classical Euler integral [23].

3.4 COMPANION FORM, COBOUNDARIES, AND THE CYCLIC VECTOR METHOD

Let a trajectory satisfy $Y_{n+1} = Y_n T(n)$ with $T(n) \in \mathrm{GL}_r(\mathbb{Q}(n))$. Fixing $e_1 = (1, 0, \ldots, 0)^\mathsf{T}$, the *Krylov* (cyclic-vector) matrix

$$U(n) = \big[e_1,\, T(n)e_1,\, \ldots,\, T(n)\cdots T(n+r-2)e_1\big]$$

converts the derivative basis into the consecutive-value basis and satisfies the *coboundary identity*

$$T(n)U(n+1) = U(n)C(n),$$

where $C(n)$ is a row *companion matrix*, i.e. the matrix of the scalar r-term recurrence obeyed by the first coordinate of Y_n. More generally, the Cyclic Vector Method [24] computes a rational gauge matrix $U(n)$ bringing any first-order matrix recurrence into companion form; the Kronecker product of two companion matrices (Section 3.5) is a typical input.

Companion coboundary matrices. We will frequently pass from higher-order scalar recurrences to first-order companion systems via the standard cyclic vector construction. The resulting gauge transformation is called the companion coboundary matrix.

Suppose the hypergeometric state row vector satisfies

$$Y_{n+1} = Y_n T(n).$$

Let $e_1 = (1, 0, \ldots, 0)^T$. Since

$$e_1,\ T(n)e_1,\ T(n)T(n+1)e_1,\ \ldots,\ T(n)\cdots T(n+r-2)e_1$$

represent the first coordinate propagated forward under the hypergeometric system, we define the companion coboundary matrix by

$$U(n) = \left(e_1,\ T(n)e_1,\ T(n)T(n+1)e_1,\ \cdots,\ T(n)\cdots T(n+r-2)e_1\right).$$

The columns of $U(n)$ therefore correspond to the successive shifts

$$F_n,\ F_{n+1},\ \ldots,\ F_{n+r-1},$$

expressed in the hypergeometric basis. Consequently, if

$$X_n = Y_n U(n),$$

then

$$X_n = (F_n, F_{n+1}, \ldots, F_{n+r-1})$$

is the companion state vector, satisfying

$$X_{n+1} = X_n C(n),$$

where

$$C(n) = U(n)^{-1} T(n) U(n+1).$$

$$\begin{tikzcd}
Y_n \arrow[r, "T(n)"] \arrow[d, "U(n)"'] & Y_{n+1} \arrow[d, "U(n+1)"] \\
X_n = (F_n, \ldots, F_{n+r-1}) \arrow[r, "C(n)"] & X_{n+1} = (F_{n+1}, \ldots, F_{n+r})
\end{tikzcd}$$

3.5 KRONECKER PRODUCTS OF COMPANION MATRICES

If f_n and g_n satisfy second-order recurrences with companion matrices M_n^f, M_n^g, then the tensor state vector $\mathbf{v}_n = (f_{n-1}g_{n-1}, f_{n-1}g_n, f_n g_{n-1}, f_n g_n)$ evolves by the Kronecker product $M_n^f \otimes M_n^g$. A cyclic-vector gauge (Section 3.4) returns this fourth-order system to companion form, so that bilinear combinations such as $c_n y_n + d_n x_n$ obey a single scalar recurrence. This is the mechanism behind the $\pi + e$ construction.

3.6 THE POINCARÉ–PERRON THEOREM

Theorem 1 (Poincaré–Perron, second-order form). *Suppose*

$$u_{n+2} = A_n u_{n+1} + B_n u_n, \qquad A_n \longrightarrow A, \quad B_n \longrightarrow B \neq 0,$$

and the roots λ_s, λ_d of $\lambda^2 - A\lambda - B = 0$ have distinct moduli. Then the recurrence has linearly independent solutions $u^{(s)}$ and $u^{(d)}$ satisfying

$$\frac{u^{(s)}_{n+1}}{u^{(s)}_n} \longrightarrow \lambda_s, \qquad \frac{u^{(d)}_{n+1}}{u^{(d)}_n} \longrightarrow \lambda_d.$$

Every nonzero solution that is eventually nonvanishing has consecutive-term ratio tending to one of these roots.

See [25]. The higher-order companion version, together with its recessive-solution consequences, is stated in Section 3.9.

3.7 PINCHERLE'S THEOREM

Proposition 1 (Pincherle). *Suppose $b_n \neq 0$ for $n \geq 1$. If*

$$Z_n = a_n Z_{n-1} + b_n Z_{n-2} \qquad (n \geq 1)$$

has a minimal solution with $Z_{-1} \neq 0$, then

$$\cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \ddots}} = -\frac{Z_0}{Z_{-1}}.$$

3.8 THE MEIJER G-FUNCTION

Definition 2 (Meijer G-function). Let $m, n, p, q \in \mathbb{N}_0$ such that $0 \leq n \leq p$ and $0 \leq m \leq q$.

The Meijer G-function [26, §16.17, Eq. (16.17.1)] is a special function defined by the following contour integral:

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, z\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s \, ds,$$

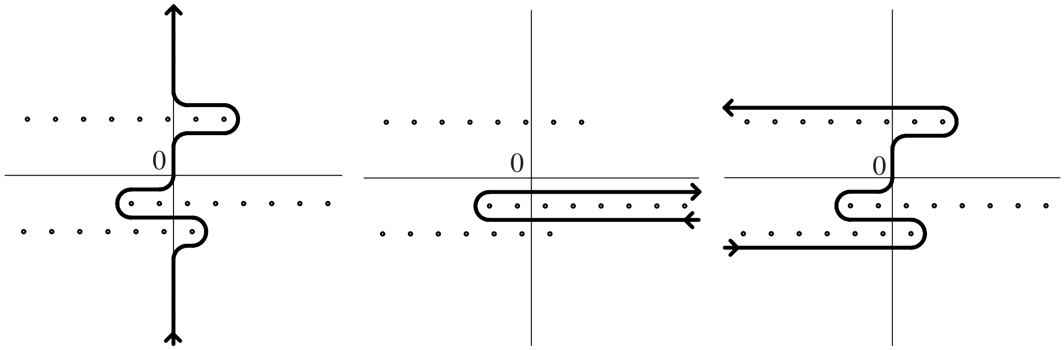
where $\Gamma(n)$ is the gamma function. The choice of contour L depends on the specific parameters of the G-function (see Fig. 1, adapted from the *NIST Digital Library of Mathematical Functions* (DLMF), Fig. 16.17.1, Release 1.2.4 (2025-03-15)). There are three options for the contour, and the definition dictates to choose the contour for which it converges (if it converges for more than one, then the different options agree):

1. L goes from $-i \cdot \infty$ to $i \cdot \infty$; the integral converges if $p + q < 2(m + n)$ and $|\arg z| < (m + n - \frac{1}{2}(p + q))\pi$.
2. L is a loop source at infinity on a line parallel to the positive real axis, encircling the poles of $\Gamma(b_\ell - s)$ once in the negative sense and returning to infinity on another line parallel to the positive real axis; the integral converges for all $z \neq 0$ if $p < q$, and for $0 < |z| < 1$ if $p = q \geq 1$.
3. L is a loop source at infinity on a line parallel to the negative real axis, encircling the poles of $\Gamma(1 - a_j + s)$ once in the positive sense and returning to infinity on another line parallel to the negative real axis; the integral converges for all z if $p > q$, and for $|z| > 1$ if $p = q \geq 1$.

The Meijer G-function is D-finite. It satisfies the following differential equation:

$$\left[(-1)^{p-m-n} \, z \prod_{j=1}^{p} \left(z \frac{d}{dz} - a_j + 1 \right) \; - \; \prod_{j=1}^{q} \left(z \frac{d}{dz} - b_j \right) \right] G^{m,n}_{p,q}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right) = 0.$$

The Mellin–Barnes integral in the above definition yields a Laplace-type representation and the contiguous operators that generate the corresponding CMF trajectory.



(a) Case (i) (b) Case (ii) (c) Case (iii)

Figure 1: Contours L in the s-plane for the integral representation of the Meijer G-function. (i) A vertical contour separating the poles of $\Gamma(b_j - s)$ and $\Gamma(1 - a_j + s)$, used when $p + q < 2(m + n)$. (ii) A contour looping around the poles of $\Gamma(b_j - s)$ (negative orientation), used when $p \le q$. (iii) A contour looping around the poles of $\Gamma(1 - a_j + s)$ (positive orientation), used when $p \ge q$. In each case, the contour is chosen so that the integral converges, and different valid choices yield the same function. Adapted from the NIST Digital Library of Mathematical Functions, Fig. 16.17.1.

3.9 Recessive (minimal) solutions and the inverse-transpose principle

Several solutions use the same recurrence-first principle: a special-function trajectory is put into companion form; an integral representation identifies it as the unique recessive solution; the inverse-transpose product then selects the vector of its initial conditions; and a rational coboundary transports this projective limit back to the matrix appearing in the question.

Definition 3 (Projective convergence)**.** Let $v_N, v \in \mathbb{C}^r \setminus \{0\}$. We write

$$[v_N] \longrightarrow [v]$$

when there are nonzero scalars γ_N such that $\gamma_N^{-1} v_N \to v$. Equivalently, whenever $v_k \neq 0$,

$$\frac{(v_N)_j}{(v_N)_k} \longrightarrow \frac{v_j}{v_k}$$

for every j for which the quotient is eventually defined.

Definition 4 (Poincaré-type recurrence and recessive solution)**.** Consider an order-r recurrence in row-companion form

$$(x_{n+1}, \dots, x_{n+r}) = (x_n, \dots, x_{n+r-1})C(n), \tag{6}$$

where

$$C(n) = C_\infty + O(n^{-1})$$

and C_∞ is invertible and diagonalizable. Suppose that C_∞ has a simple root λ_* such that

$$0 < |\lambda_*| < \min\{|\lambda| : \lambda \in \operatorname{spec}(C_\infty),\ \lambda \neq \lambda_*\}. \tag{7}$$

A nonzero solution $F = (F_n)$ is called *recessive* if its state vectors

$$\mathbf{F}_n = (F_n, F_{n+1}, \dots, F_{n+r-1})^\mathsf{T}$$

span the one-dimensional solution associated with λ_*. In particular, its exponential rate is $|\lambda_*|$, whereas every solution not on that line has strictly larger exponential rate. The terms *minimal solution* and *recessive solution* are synonymous here.

The other characteristic roots need not have different moduli. Only the one-dimensional root in (7) must be separated from the complementary block.

Lemma 2 (Recognition from a root-rate bound)**.** *Assume* (7)*. Let F_n be a nonzero solution of* (6)*. If*

$$\limsup_{n\to\infty} |F_n|^{1/n} \le |\lambda_*|,$$

then F spans the unique recessive solution line.

Proof. The asymptotic factorization in [19, Lemma 4.3 and Corollary 4.4] is stated when all characteristic-root moduli are distinct. Its proof first uses an asymptotic coboundary to separate the limiting spectral subspaces and then applies the discrete Levinson theorem of Benzaid–Lutz. The same proof, with the decomposition

$$\mathbb{C}^r = E_* \oplus E_{\mathrm{comp}},$$

uses only the gap between the one-dimensional λ_*-space and the complementary spectral block; no separation inside E_{comp} is needed. It follows that there is a unique one-dimensional solution of rate $|\lambda_*|$, while every solution outside it has exponential rate at least the smallest modulus in the complementary block. The displayed root bound therefore forces F to lie on the recessive line. $\square$

Corollary 1 (Inverse transpose selects the recessive initial vector)**.** *Let F be the recessive solution from Lemma 2, and put*

$$\mathbf{F}_0 = (F_0, F_1, \ldots, F_{r-1})^{\mathsf{T}}, \qquad \Phi_C(N) = C(0)C(1)\cdots C(N-1).$$

Let w_N be terminal vectors having nonzero asymptotic connection with the recessive dual mode. Then

$$[\Phi_C(N)^{-\mathsf{T}} w_N] \longrightarrow [\mathbf{F}_0]. \tag{8}$$

For a fixed standard column $w_N = e_j$, the hypothesis is the nonvanishing of one connection coefficient. For the final standard column of a genuine row companion matrix, this coefficient is automatically nonzero.

Proof. In the strict setting of [19, Lemma 4.3], write the factorization of the fundamental product as

$$\Phi_C(N) = B\ \mathrm{diag}\big(\lambda_i^N N^{\gamma_i}\big)(I + o(1))A.$$

Taking inverse transposes replaces every characteristic factor by its reciprocal. Since $|\lambda_*|$ is uniquely smallest, its reciprocal is uniquely largest. Thus every terminal vector with a nonzero coefficient in that dual direction is sent projectively to $B^{-\mathsf{T}} e_*$. The forward product applied to this vector has rate $|\lambda_*|$, so $B^{-\mathsf{T}} e_*$ is, up to scale, exactly the initial-condition vector $\mathbf{F}_0$ of the recessive solution. In the isolated-root setting, the same calculation is made after replacing the individual nonrecessive modes by the single complementary block described in the proof of Lemma 2.

This is the matrix form of backward recurrence, commonly called Miller's algorithm; see [27, 28]. $\square$

Corollary 2 (Transport through a rational coboundary)**.** *Let $T(n), C(n) \in \mathrm{GL}_r(\mathbb{Q}(n))$ and let $U(n) \in \mathrm{GL}_r(\mathbb{Q}(n))$ satisfy*

$$T(n)U(n+1) = U(n)C(n). \tag{9}$$

Define

$$\Phi_T(N) = T(0)\cdots T(N-1), \qquad \Phi_C(N) = C(0)\cdots C(N-1).$$

Then

$$\Phi_T(N) = U(0)\Phi_C(N)U(N)^{-1} \tag{10}$$

and hence

$$\Phi_T(N)^{-\mathsf{T}} = U(0)^{-\mathsf{T}}\Phi_C(N)^{-\mathsf{T}}U(N)^{\mathsf{T}}. \tag{11}$$

Suppose F is the recessive companion solution and that $U(N)^{\mathsf{T}} e_j$ has nonzero connection with the recessive dual mode. Then

$$[\Phi_T(N)^{-\mathsf{T}} e_j] \longrightarrow [U(0)^{-\mathsf{T}}\mathbf{F}_0]. \tag{12}$$

Proof. Multiplying (9) for $n = 0, \ldots, N-1$ makes all intermediate $U(n)$ cancel and gives (10). Taking inverse transposes gives (11). Rational matrices and their inverses have at most polynomial growth along the integers away from their finitely many poles. Such factors cannot change an exponentially isolated one-dimensional mode. Applying Corollary 1 to the terminal vector $U(N)^{\mathsf{T}} e_j$ proves (12). This is the trajectory-level form of coboundary equivalence used throughout [19]. $\square$

Corollary 3 (Invariant pairings and ratios of dual solutions). *Let $x_{n+1} = A(n)x_n$ have a unique recessive solution x_n, and let the dual evolution be*

$$\ell_{n+1}^{\mathsf{T}} = \ell_n^{\mathsf{T}} A(n)^{-1}.$$

Then the pairing $\ell_n^{\mathsf{T}} x_n$ is independent of n. Moreover, the inverse-transpose selection in Corollary 1 implies that the dual system has a unique dominant projective direction. Consequently, let $\ell^{(p)}$ and $\ell^{(q)}$ be two dual solutions with nonzero invariant pairings with x. If the jth coordinate is nonzero on the common dominant dual direction, then

$$\frac{(\ell_n^{(p)})_j}{(\ell_n^{(q)})_j} \longrightarrow \frac{(\ell_0^{(p)})^{\mathsf{T}} x_0}{(\ell_0^{(q)})^{\mathsf{T}} x_0}.$$

The same statement holds after rational coboundaries and finite index shifts, with the initial pairing transported by the corresponding exact matrices.

Proof. The identity

$$\ell_{n+1}^{\mathsf{T}} x_{n+1} = \ell_n^{\mathsf{T}} A(n)^{-1} A(n) x_n = \ell_n^{\mathsf{T}} x_n$$

proves conservation. The dual dominant line is the reciprocal of the primal recessive line by Corollary 1. Thus the leading coefficient of a dual solution is a nonzero common normalization times its conserved pairing with x. Taking the quotient cancels the common normalization. Rational coboundaries preserve this statement by Corollary 2. □

Remark 1 (Projectively irrelevant scalar factors). If the matrix in a challenge satisfies

$$W(n) = \sigma(n) T(n)^{-\mathsf{T}}, \qquad \sigma(n) \neq 0,$$

then

$$W(0) \cdots W(N-1) = \left(\prod_{n=0}^{N-1} \sigma(n) \right) \Phi_T(N)^{-\mathsf{T}}.$$

The scalar product has no effect on projective limits or on ratios of two fixed rows. This is also where a stepwise sign may be absorbed.

3.10 COMMON PROOF PATTERN FOR QUESTIONS 3 AND 5–8.

For each recessive-solution question the proof has the same shape:

(i) construct a special-function trajectory $T(n)$ and verify the exact inverse-transpose identification with the challenge matrix;

(ii) construct a rational companion coboundary and verify that the concrete sequence F_n satisfies the resulting scalar recurrence;

(iii) compute the limiting characteristic polynomial and isolate its unique recessive solution;

(iv) use an integral representation and a Laplace/root-rate argument to identify F_n as the recessive solution;

(v) evaluate the appropriate linear functionals on the initial-condition vector $(F_0, \ldots, F_{r-1})^{\mathsf{T}}$.

3.11 READING THE DEPENDENCY GRAPHS.

Each solution ends with the dependency graph of its Lean-style blueprint. An arrow $X \to Y$ means that item Y is proved using item X. Rounded gray boxes are definitions, blue boxes are lemmas, the thick green box is the goal theorem, and dashed orange boxes are items imported from these Preliminaries.

3.12 CODE

All supporting code described here is available in our GitHub repository.[1]

[1] https://github.com/RamanujanMachine/RamanujanChallengeSolutions

4 POLYNOMIAL CONTINUED FRACTION FOR π

This section proves Question 1. The formula was discovered as the trajectory $(1,1,3)$ in the Conservative Matrix Field of Section 3.1, which is generated from the ${}_2F_1$ hypergeometric function (that was shown to be strongly related to π in Section 5 of [22]) at argument $w = 1/2$ [18, 19, 22]. The proof follows the mechanism used for trajectory $(1,1,2)$ in Appendix G.4 of [22], with parameters and normalization changed for $(1,1,3)$.

4.1 PROOF STRATEGY

The proof has five steps:

1. Choose the trajectory generating the formula and compute its trajectory matrix $T(n)$.
2. Derive from the state-vector $[F_n, G_n]$ a consecutive-value vector $[F_n, F_{n+1}]$, obtaining a scalar recurrence.
3. Rescale that recurrence to the polynomial coefficients a_n, b_n in Equation (1).
4. Use Euler's integral (Lemma 1) and the Poincaré–Perron theorem (Theorem 1) to prove that the hypergeometric solution follows the smaller characteristic root and is therefore minimal.
5. Transport the minimal solution to the polynomial recurrence and use Pincherle's theorem (Proposition 1), with exact values of F_1 and F_2, to evaluate the continued fraction.

Appendix G.4 of [22] supplies the central convergence pattern: put the trajectory matrix in companion form, compute the two limiting characteristic roots, and bound Euler's integral sharply enough to identify the root followed by the hypergeometric solution. The present proof adds the explicit scaling to the challenge PCF and the final Pincherle evaluation.

4.2 THE TRAJECTORY AND ITS SCALAR RECURRENCE

We use the ${}_2F_1$ CMF state-vector and basis matrices of Section 3.1. Set the starting point and trajectory direction

$$p = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \qquad v = (1,1,3),$$

and define

$$F_n = {}_2F_1\left(n+\tfrac{1}{2}, n+\tfrac{1}{2}; 3n+\tfrac{1}{2}; \tfrac{1}{2}\right), \qquad G_n = \Theta\, {}_2F_1\left(n+\tfrac{1}{2}, n+\tfrac{1}{2}; 3n+\tfrac{1}{2}; w\right)\Big|_{w=1/2}. \tag{13}$$

Lemma 3 (Scalar recurrence)**.** *For*

$$B(n) = \frac{9(5n+6)(6n+1)(6n+5)(6n+7)(6n+11)}{16(n+1)^2(2n+3)^2(5n+1)}, \tag{14}$$

$$A(n) = -\frac{3(6n+7)(6n+11)(220n^3+484n^2+301n+42)}{8(n+1)^2(2n+3)^2(5n+1)}, \tag{15}$$

the sequence F_n of Equation (13) *satisfies*

$$F_{n+2} = A(n)F_{n+1} + B(n)F_n, \qquad n \geq 1, \tag{16}$$

Proof. At $p + nv = (n+1/2, n+1/2, 3n+1/2)$, a single step in the direction v uses the trajectory matrix

$$\begin{aligned} T(n) = {} & M_x(x_n, y_n, z_n)M_y(x_n+1, y_n, z_n) \\ & \times M_z(x_n+1, y_n+1, z_n)M_z(x_n+1, y_n+1, z_n+1) \\ & \times M_z(x_n+1, y_n+1, z_n+2), \end{aligned} \tag{17}$$

where $(x_n, y_n, z_n) = p + nv$. Multiplication and factorization give

$$T(n) = \begin{pmatrix} -\dfrac{3(n+1)(6n+1)(6n+5)}{8n^2(2n+1)} & \dfrac{3(6n+1)(6n+5)(10n^2+15n+6)}{16n^2(2n+1)} \\[2ex] \dfrac{3(5n+1)(6n+1)(6n+5)}{4n^2(2n+1)^2} & -\dfrac{9(6n+1)(6n+5)(14n^2+13n+2)}{8n^2(2n+1)^2} \end{pmatrix}. \tag{18}$$

$T(n)$ shifts the "state-vector":

$$[F_n, G_n]T(n) = [F_{n+1}, G_{n+1}].$$

Following Section 3.4, define

$$U(n) = \begin{pmatrix} 1 & -\dfrac{3(n+1)(6n+1)(6n+5)}{8n^2(2n+1)} \\ 0 & \dfrac{3(5n+1)(6n+1)(6n+5)}{4n^2(2n+1)^2} \end{pmatrix}. \tag{19}$$

The first column of Equation (18) shows directly that $[F_n, G_n]U(n) = [F_n, F_{n+1}]$, so $U(n)$ changes from the derivative basis used by the CMF to the consecutive-value basis used by a second-order recurrence. The same forward shift must agree in both bases, giving the coboundary identity

$$T(n)U(n+1) = U(n)C(n), \qquad C(n) = \begin{pmatrix} 0 & B(n) \\ 1 & A(n) \end{pmatrix}. \tag{20}$$

The scalar recurrence for the hypergeometric sequence now follows by the coboundary identity (20) applied to $[F_n, F_{n+1}]$. ☐

4.3 Normalization to the challenge PCF

Set

$$r_n = \frac{8(n+1)^2(2n+3)^2(5n+1)}{3(6n+7)(6n+11)}. \tag{21}$$

Substitution into Equations (14) and (15) gives the exact identities

$$r_n A(n) = a_n, \qquad r_n r_{n-1} B(n) = b_n, \tag{22}$$

with a_n, b_n as in Equation (1). Choose

$$s_0 = 1, \qquad s_{-1} = \frac{77}{24}, \qquad \frac{s_n}{s_{n-1}} = r_n \quad (n \geq 0), \tag{23}$$

with $r_0 = 24/77$. Define

$$X_n = s_n F_{n+2} \qquad (n \geq -1). \tag{24}$$

This rescaling carries the hypergeometric recurrence to the challenge recurrence.

Lemma 4 (Rescaled recurrence)**.** *With $X_n = s_n F_{n+2}$ as in Equation* (24)*, the sequence X_n satisfies the challenge recurrence*

$$X_n = a_n X_{n-1} + b_n X_{n-2}, \qquad n \geq 1, \tag{25}$$

where a_n, b_n are the coefficients of Equation (1)*.*

Proof. Multiply Equation (16) by s_n and use the transport identities (22) together with the scaling (23). ☐

4.4 The Euler-integral minimal solution

Lemma 5. *F_n is a minimal solution of Equation* (16)*: there exists a linearly independent solution D_n such that $F_n/D_n \to 0$.*

Proof. The companion matrix has the limit

$$C(n) \longrightarrow C_\infty = \begin{pmatrix} 0 & 729/4 \\ 1 & -297/2 \end{pmatrix}. \tag{26}$$

The characteristic equation $\lambda^2 + (297/2)\lambda - 729/4 = 0$ has roots

$$\lambda_s = \frac{-297 + 135\sqrt{5}}{4} > 0, \qquad \lambda_d = \frac{-297 - 135\sqrt{5}}{4} < 0, \tag{27}$$

and $0 < \lambda_s < |\lambda_d|$.

Euler's integral (Lemma 1) applies for every integer $n \geq 1$ because $3n + 1/2 > n + 1/2 > 0$. It gives

$$F_n = \frac{\Gamma(3n+\frac{1}{2})}{\Gamma(n+\frac{1}{2})\Gamma(2n)} \int_0^1 t^{n-\frac{1}{2}}(1-t)^{2n-1}\left(1-\frac{t}{2}\right)^{-n-\frac{1}{2}} dt. \tag{28}$$

Set

$$h(t) = \frac{t(1-t)^2}{1-t/2}, \qquad \psi(t) = t^{-1/2}(1-t)^{-1}(1-t/2)^{-1/2}.$$

The integral in Equation (28) is $\int_0^1 h(t)^n \psi(t)\, dt$. Differentiation gives

$$h'(t) = -\frac{4(t-1)(t^2-3t+1)}{(t-2)^2},$$

so the unique maximum in $(0,1)$ occurs at

$$t_* = \frac{3-\sqrt{5}}{2}, \qquad h_* := h(t_*) = -11 + 5\sqrt{5}.$$

The singular factor $\psi(t)$ at $t = 1$ requires an endpoint estimate. On $[0, 3/4]$, ψ is integrable and $0 \leq h(t) \leq h_*$, so

$$\int_0^{3/4} h(t)^n \psi(t)\, dt \leq C_0 h_*^n, \qquad C_0 = \int_0^{3/4} \psi(t)\, dt < \infty. \tag{29}$$

On $[3/4, 1]$, return to the original integrand. For $n \geq 1$, $t^{n-\frac{1}{2}} \leq 1$ and $(1-t/2)^{-n-\frac{1}{2}} \leq 2^{n+\frac{1}{2}}$, so

$$\begin{aligned}&\int_{3/4}^1 t^{n-\frac{1}{2}}(1-t)^{2n-1}\left(1-\frac{t}{2}\right)^{-n-\frac{1}{2}} dt \\ &\qquad \leq 2^{n+\frac{1}{2}} \int_{3/4}^1 (1-t)^{2n-1}\, dt = \frac{\sqrt{2}}{2n}\left(\frac{1}{8}\right)^n. \end{aligned}\tag{30}$$

Since $1/8 < h_*$, Equations (29) and (30) give

$$\int_0^1 t^{n-\frac{1}{2}}(1-t)^{2n-1}\left(1-\frac{t}{2}\right)^{-n-\frac{1}{2}} dt = O(h_*^n). \tag{31}$$

Stirling's formula gives

$$\frac{\Gamma(3n+\frac{1}{2})}{\Gamma(n+\frac{1}{2})\Gamma(2n)} = \sqrt{\frac{n}{\pi}}\left(\frac{27}{4}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right). \tag{32}$$

Combining Equations (28), (31) and (32),

$$F_n = O\Big(\sqrt{n}\,\left[\tfrac{27}{4}(-11+5\sqrt{5})\right]^n\Big) = O(\sqrt{n}\,\lambda_s^n).$$

The hypotheses of Theorem 1 hold because Equations (26) and (27) give finite limiting coefficients, $B = 729/4 \neq 0$, and roots of distinct moduli. The Euler integrand is positive, so $F_n > 0$. The bound above excludes growth with modulus $|\lambda_d|$: if $F_{n+1}/F_n \to \lambda_d$, then $|F_n|^{1/n} \to |\lambda_d|$, whereas the proved bound gives $\limsup_{n\to\infty} |F_n|^{1/n} \leq \lambda_s < |\lambda_d|$. Therefore

$$\frac{F_{n+1}}{F_n} \longrightarrow \lambda_s.$$

Choose a solution D_n associated with λ_d. The quotient then satisfies

$$\left|\frac{F_n}{D_n}\right|^{1/n} \longrightarrow \left|\frac{\lambda_s}{\lambda_d}\right| < 1,$$

so $F_n/D_n \to 0$. □

Lemma 6. *The sequence X_n from Equation* (24) *is a minimal solution of Equation* (25)*.*

Proof. If $Y_n = s_n D_{n+2}$, the same calculation based on Equation (22) shows that Y_n satisfies Equation (25). Moreover $X_n/Y_n = F_{n+2}/D_{n+2} \to 0$, so the common scaling s_n preserves minimality. □

4.5 Pincherle evaluation and the two anchor integrals

We evaluate the two anchor values F_1 and F_2.

Lemma 7 (Anchor values)**.** *The first two hypergeometric values are*

$$F_1 = \frac{15\sqrt{2}}{2}(\pi - 3), \qquad F_2 = \frac{1155\sqrt{2}}{8}(22 - 7\pi). \tag{33}$$

Proof. Euler's integral (Lemma 1) gives

$$F_1 = \frac{15}{4}\int_0^1 t^{1/2}(1-t)(1-t/2)^{-3/2}\,dt, \tag{34}$$

$$F_2 = \frac{1155}{32}\int_0^1 t^{3/2}(1-t)^3(1-t/2)^{-5/2}\,dt. \tag{35}$$

Use $t = 2\sin^2\theta$, $u = \tan\theta$, $0 \le u \le 1$, making Equations (34) and (35)

$$F_1 = 15\sqrt{2}\int_0^1 \frac{u^2(1-u^2)}{(1+u^2)^2}\,du, \tag{36}$$

$$F_2 = \frac{1155\sqrt{2}}{4}\int_0^1 \frac{u^4(1-u^2)^3}{(1+u^2)^4}\,du. \tag{37}$$

The required decompositions are

$$\frac{u^2(1-u^2)}{(1+u^2)^2} = -1 + \frac{3}{1+u^2} - \frac{2}{(1+u^2)^2},$$

$$\frac{u^4(1-u^2)^3}{(1+u^2)^4} = -u^2 + 7 - \frac{25}{1+u^2} + \frac{38}{(1+u^2)^2} - \frac{28}{(1+u^2)^3} + \frac{8}{(1+u^2)^4}.$$

Let $I_k = \int_0^1 (1+u^2)^{-k}\,du$. Integration by parts gives

$$I_1 = \frac{\pi}{4}, \qquad I_k = \frac{2k-3}{2k-2}I_{k-1} + \frac{1}{2^k(k-1)} \quad (k \ge 2).$$

Substitution in the two decompositions gives

$$\int_0^1 \frac{u^2(1-u^2)}{(1+u^2)^2}\,du = \frac{\pi-3}{2}, \qquad \int_0^1 \frac{u^4(1-u^2)^3}{(1+u^2)^4}\,du = 11 - \frac{7\pi}{2}.$$

□

Theorem 2 (Challenge continued fraction)**.** *The polynomial continued fraction of Question 1 evaluates to*

$$\mathrm{PCF}(a_n, b_n) = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \ddots}} = \frac{6}{3-\pi},$$

proving Equation (2)*.*

Proof. By Equations (23) and (24) and Lemma 7,

$$X_{-1} = \frac{77}{24}F_1 \neq 0, \qquad X_0 = F_2.$$

Lemma 4 makes X_n a solution of the challenge recurrence and Lemma 6 makes it minimal. Since $b_n \neq 0$ for every $n \ge 1$, Pincherle's theorem (Proposition 1) applies and gives

$$\cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \ddots}} = -\frac{X_0}{X_{-1}} = -\frac{24}{77}\frac{F_2}{F_1} = \frac{6(22-7\pi)}{3-\pi}.$$

Using $a_0 = -42$,

$$\mathrm{PCF}(a_n, b_n) = -42 + \frac{6(22-7\pi)}{3-\pi} = \boxed{\frac{6}{3-\pi}}.$$

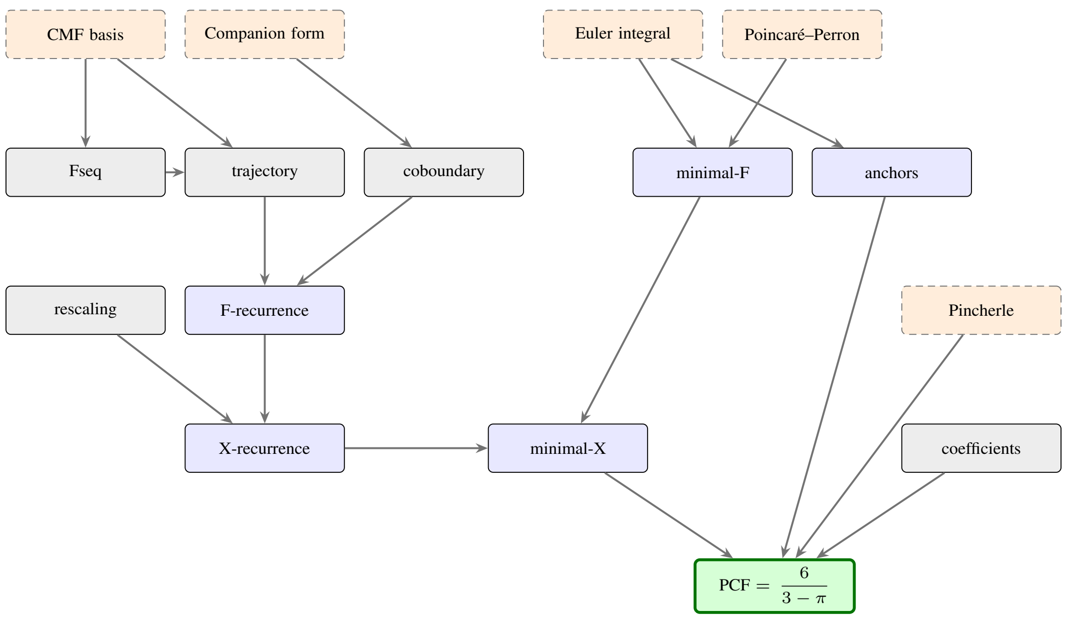
□

Note: The companion roots also predict the asymptotic decimal convergence rate (the error is asymptotically proportional to F_n/D_n):

$$\log_{10}\left|\frac{\lambda_d}{\lambda_s}\right| = \log_{10}\left(\frac{123+55\sqrt{5}}{2}\right) \approx 2.09$$

digits per continued-fraction level.

4.6 Blueprint dependency graph



5 Euler's constant γ as an Apéry limit

This section proves Question 2. It is a specific case of the main result of the upcoming paper "An Infinite Family of Apéry-like Recurrences for Euler's Constant".

5.1 Proof strategy

The recurrence in the question is the companion form of a rank-three Meijer G-function trajectory (Definition 2). The proof runs forward in four steps, realized below as Lemmas 8 to 11 and Theorem 3.

1. Two *initial identities* (Lemma 8) express γ and 1 as fixed pairings of two covectors ℓ_γ, ℓ_1 with the initial state vector $S(0)$.
2. A Mellin–Barnes saddle-point estimate (Lemma 9) shows that the Meijer G state vector $S(n)$ is the *unique recessive* solution of the trajectory system (Lemma 10).
3. A rational gauge (Section 3.4) turns the trajectory into the companion recurrence of the question and carries ℓ_γ, ℓ_1 to its initial rows; this *identifies the question's sequences* as $p_n = \ell_\gamma^{\mathsf{T}} U(0)\Phi_C(n+1)e_3$ and $q_n = \ell_1^{\mathsf{T}} U(0)\Phi_C(n+1)e_3$ (Lemma 11).
4. The inverse-transpose / Miller selection principle (Corollary 1) sends the dual product to $S(0)$, so $p_n/q_n \to \ell_\gamma^{\mathsf{T}} S(0)/\ell_1^{\mathsf{T}} S(0) = \gamma$ (Theorem 3).

5.2 Proof

Throughout, let $\theta = z\frac{d}{dz}$ such that $\theta F(c)$ denotes $\theta F(z)\big|_{z=c}$.

Define

$$H_n(z) = G^{3,2}_{2,3}\left(\begin{matrix} 1,1 \\ 1+n, 1+n, 1+n \end{matrix}\,\middle|\, z\right)$$

and let

$$S(n)^{\mathsf{T}} = \big(H_n(1), \theta H_n(1), \theta^2 H_n(1)\big)$$

be the Meijer G-function state vector.

The Meijer G-function contiguous relations produce a matrix $M(n)$ with $S(n+1)^{\mathsf{T}} = S(n)^{\mathsf{T}} M(n)$, explicitly

$$M(n) = \begin{bmatrix} 0 & (n+1)^3 & (n+1)^3\,(3n+4) \\ 0 & -3\,(n+1)^2 & (-8n-11)\,(n+1)^2 \\ 1 & 3n+3 & 6n^2+12n+5 \end{bmatrix}.$$

Fix the constant covectors

$$\ell_\gamma = (0,1,0)^{\mathsf{T}}, \qquad \ell_1 = (1,-1,1)^{\mathsf{T}}, \qquad e_3 = (0,0,1)^{\mathsf{T}},$$

and set $\Phi_M(N) = M(0)M(1)\cdots M(N-1)$.

Step 1: γ and 1 as pairings with $S(0)$.

Lemma 8. *Let*

$$H(z) := H_0(z) = G^{3,2}_{2,3}\left(\begin{matrix} 1,1 \\ 1,1,1 \end{matrix} \,\middle|\, z \right), \qquad \theta = z\frac{d}{dz}.$$

Then

$$\theta H(1) = \gamma, \qquad (1-\theta+\theta^2)H(1) = 1.$$

Proof. By the Mellin–Barnes definition of the Meijer G-function [26, §16.17, Eq. (16.17.1)], together with Euler's reflection formula [26, §5.5(ii), Eq. (5.5.3)], we have

$$H(z) = \frac{1}{2\pi i}\int_L \Gamma(s+1)^3\Gamma(-s)^2 z^{-s}\,ds = \frac{1}{2\pi i}\int_L \Gamma(s+1)\pi^2\csc^2(\pi s) z^{-s}\,ds.$$

Using

$$\pi^2\csc^2(\pi s) = \int_0^\infty x^{-s-1}\frac{\log x}{x-1}\,dx$$

initially in the common strip of convergence and then by meromorphic continuation, Mellin inversion gives

$$H(z) = \int_0^\infty \left(\frac{1}{2\pi i}\int_L \Gamma(s+1)(zx)^{-s}\,ds\right)\frac{\log x}{x}\frac{dx}{x-1} = \int_0^\infty ze^{-zx}\frac{\log x}{x-1}\,dx.$$

Differentiating under the integral sign, we compute the two pairings directly:

$$\theta H(1) = \int_0^\infty \theta\left(ze^{-zx}\right)\big|_{z=1}\frac{\log x}{x-1}\,dx = \int_0^\infty e^{-x}(1-x)\frac{\log x}{x-1}\,dx = -\int_0^\infty e^{-x}\log x\,dx = -\Gamma'(1) = \gamma,$$

where we used

$$\theta\left(ze^{-zx}\right)\big|_{z=1} = e^{-x}(1-x), \qquad \Gamma'(1) = -\gamma.$$

Similarly,

$$\begin{aligned}(1-\theta+\theta^2)H(1) &= \int_0^\infty (1-\theta+\theta^2)\left(ze^{-zx}\right)\big|_{z=1}\frac{\log x}{x-1}\,dx = \int_0^\infty e^{-x}(x-1)^2\frac{\log x}{x-1}\,dx \\ &= \int_0^\infty e^{-x}(x-1)\log x\,dx = \Gamma'(2)-\Gamma'(1) = (1-\gamma)-(-\gamma) = 1,\end{aligned}$$

using

$$(1-\theta+\theta^2)\left(ze^{-zx}\right)\big|_{z=1} = e^{-x}(x-1)^2, \qquad \Gamma'(2) = 1-\gamma.$$

□

Because $H = H_0$ and $S(0)^{\mathsf{T}} = (H_0(1), \theta H_0(1), \theta^2 H_0(1))$, the two identities of Lemma 8 are exactly

$$\ell_\gamma^{\mathsf{T}} S(0) = \gamma, \qquad \ell_1^{\mathsf{T}} S(0) = 1. \tag{38}$$

Step 2: $S(n)$ is the unique recessive solution.

Lemma 9. *Let*

$$H_n(z) = G^{3,2}_{2,3}\left(\begin{matrix} 1,1 \\ n+1, n+1, n+1 \end{matrix} \middle| z\right), \qquad \theta = z\frac{d}{dz}.$$

Then, for $k = 0, 1, 2$,

$$\theta^k H_n(1) = \Gamma(n+1)^2 n^k \exp\left(-3n^{2/3} + O(n^{1/3})\right),$$

equivalently

$$\big(H_n(1), \theta H_n(1), \theta^2 H_n(1)\big) = \Gamma(n+1)^2 \exp\left(-3n^{2/3} + O(n^{1/3})\right) \big(1, n, n^2\big).$$

Proof. We estimate the growth of this Meijer G-solution by applying the saddle-point method to its Mellin–Barnes integral.

By the Mellin–Barnes representation (Definition 2),

$$\theta^k H_n(1) = \frac{1}{2\pi i}\int_L s^k \Gamma(n+1-s)^3 \Gamma(s)^2 \, ds,$$

where L separates the poles of $\Gamma(s)$ from those of $\Gamma(n+1-s)$. Put $x = n+1-s$, so

$$\theta^k H_n(1) = \frac{1}{2\pi i}\int_{L'} (n+1-x)^k \Gamma(x)^3 \Gamma(n+1-x)^2 \, dx.$$

The exponential part of the integrand is governed by $\Phi_n(x) = 3\log\Gamma(x) + 2\log\Gamma(n+1-x)$, with saddle equation $\Phi_n'(x) = 3\psi(x) - 2\psi(n+1-x) = 0$. Using $\psi(x) = \log x + O(1/x)$,

$$3\log x - 2\log(n+1-x) = O(n^{-2/3})$$

near the relevant saddle, so $\xi_n = n^{2/3} + O(n^{1/3})$. Taylor expansion of $\log\Gamma(n+1-\xi_n)$ around $n+1$ with Stirling's formula [26, §5.11(i), Eq. (5.11.1)] for $\log\Gamma(\xi_n)$ gives

$$\Phi_n(\xi_n) = 2\log\Gamma(n+1) - 3n^{2/3} + O(n^{1/3}),$$

Moreover,

$$\Phi_n''(\xi_n) = 3\psi'(\xi_n) + 2\psi'(n+1-\xi_n) = 3n^{-2/3} + O(n^{-1}).$$

The second-order approximation of the phase at the saddle therefore contributes only a polynomial prefactor, which is absorbed into $\exp(O(n^{1/3}))$.

Finally $(n+1-x)^k = n^k(1 + O(n^{-1/3}))$ along the saddle for $k = 0, 1, 2$, and the standard saddle-point estimate gives

$$\theta^k H_n(1) = n^k \exp\left(\Phi_n(\xi_n) + O(\log n)\right) = \Gamma(n+1)^2 n^k \exp\left(-3n^{2/3} + O(n^{1/3})\right).$$

□

Lemma 9 pins $S(n)$ down among all solutions of the trajectory system.

Lemma 10. *$S(n)^\mathsf{T}$ is the unique recessive solution (Definition 4) of the trajectory system $S(n+1)^\mathsf{T} = S(n)^\mathsf{T} M(n)$.*

Proof. By Lemma 9,

$$S(n) = \Gamma(n+1)^2 \exp\left(-3n^{2/3} + O(n^{1/3})\right)\begin{pmatrix} 1 \\ n \\ n^2 \end{pmatrix},$$

so, by Stirling's formula [26, §5.11(i), Eq. (5.11.3)] $\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, $S(n)$ has exponential scale

$$e^{-2n} n^{2n+O(1)} \exp\left(-3n^{2/3} + O(n^{1/3})\right).$$

Applying the Cyclic Vector Method (Section 3.4, [24]) converts the rank-three system defined by $M(n)$ into a scalar third-order recurrence. The Birkhoff–Trjitzinsky method [29] [30, Ch. 7, §7.3] computes the formal asymptotic scales of its solutions; in our case this was implemented in `ramanujantools` [21] and checked against RISC's *Asymptotics* package [31]. The three scales are

$$e^{n^{1/3}-3n^{2/3}-2n}n^{2n+19/3}, \quad e^{-(-1)^{1/3}n^{1/3}-3(-1)^{2/3}n^{2/3}-2n}n^{2n+19/3}, \quad e^{(-1)^{2/3}n^{1/3}+3(-1)^{1/3}n^{2/3}-2n}n^{2n+19/3}.$$

Taking principal branches,

$$(-1)^{1/3}=e^{\pi i/3}, \qquad (-1)^{2/3}=e^{2\pi i/3},$$

the first scale has real exponential part

$$-3n^{2/3}+O(n^{1/3}),$$

whereas the two conjugate scales have real exponential part

$$\frac{3}{2}n^{2/3}+O(n^{1/3}).$$

Thus the third-order system has a one-dimensional recessive asymptotic subspace. The saddle-point estimate above shows that $S(n)^{\mathsf{T}}$ has exactly this recessive scale. Hence $S(n)^{\mathsf{T}}$ spans the unique recessive solution of the trajectory system.

□

Step 3: the companion recurrence and the question's sequences.

The next lemma exhibits the gauge that turns the trajectory into the companion recurrence of Question 2 and writes p_n and q_n in the companion basis. Its limit is evaluated in Theorem 3.

Lemma 11. *Let*

$$\widehat{M}(n)=(n+1)^4M(n)^{-\mathsf{T}}=\begin{bmatrix} 6n^2+21n+18 & -8n-11 & 3 \\ 3n^3+12n^2+16n+7 & -3n^2-7n-4 & n+1 \\ (n+1)^4 & 0 & 0 \end{bmatrix}.$$

be the denominator-cleared dual matrix of $M(n)$.

a) The gauge matrix for $\widehat{M}(n)$ is

$$U(n)=\begin{bmatrix} 1 & (2n+3)(3n+6) & (n+2)\left(15n^3+117n^2+286n+220\right) \\ 0 & (n+1)\left(3n^2+9n+7\right) & (n+1)\left(10n^4+86n^3+267n^2+358n+179\right) \\ 0 & (n+1)^4 & (n+1)^4(2n+5)(3n+9) \end{bmatrix},$$

b) Let $C(n)=U(n)^{-1}\widehat{M}(n)U(n+1)$ be the row companion matrix of the recurrence in Question 2. For $R(n)=(u_{n-3},u_{n-2},u_{n-1})^{\mathsf{T}}$, the identity $R(n+1)^{\mathsf{T}}=R(n)^{\mathsf{T}}C(n)$ is equivalent to

$$\begin{aligned} 0 = {} & \left(-8n^3-51n^2-105n-68\right)u_n \\ & +\left(24n^5+337n^4+1833n^3+4818n^2+6092n+2928\right)u_{n-1} \\ & -(n+2)(n+3)\left(24n^5+273n^4+1150n^3+2154n^2+1635n+268\right)u_{n-2} \\ & +(n+1)(n+2)^4(n+3)\left(8n^3+75n^2+231n+232\right)u_{n-3}. \end{aligned}$$

c) The gauged covectors reproduce the initial data of the question,

$$\begin{bmatrix} \ell_\gamma^{\mathsf{T}} \\ \ell_1^{\mathsf{T}} \end{bmatrix} U(0)=\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} U(0)=\begin{bmatrix} 0 & 7 & 179 \\ 1 & 12 & 306 \end{bmatrix},$$

d) *Writing* $\Phi_C(N) = C(0)C(1)\cdots C(N-1)$,

$$(p_{-3}, p_{-2}, p_{-1}) = \ell_\gamma^{\mathsf{T}} U(0), \qquad (q_{-3}, q_{-2}, q_{-1}) = \ell_1^{\mathsf{T}} U(0),$$

Thus, the two solutions of Question 2 are, in companion coordinates,

$$p_n = \ell_\gamma^{\mathsf{T}} U(0)\, \Phi_C(n+1)\, e_3, \qquad q_n = \ell_1^{\mathsf{T}} U(0)\, \Phi_C(n+1)\, e_3. \tag{39}$$

Proof. a) $U(n)$ is obtained by applying the Cyclic Vector Method (Section 3.4, [24]); the scalar factor $(n+1)^4$ is projectively irrelevant (Remark 1).

b) By the definition of row companion form in Section 3.4, the identity $R(n+1)^{\mathsf{T}} = R(n)^{\mathsf{T}} C(n)$, with $R(n)^{\mathsf{T}} = (u_{n-3}, u_{n-2}, u_{n-1})$, is equivalent to the displayed scalar recurrence. This is verified by substituting $C(n) = U(n)^{-1}\widehat{M}(n)U(n+1)$ and comparing the last column.

c) The stated product $\left[\ell_\gamma^{\mathsf{T}}; \ell_1^{\mathsf{T}}\right] U(0) = \left[0, 7, 179;\ 1, 12, 306\right]$ is a direct evaluation, and its two rows are exactly the initial data of Question 2.

d) Because $C(n)$ is a row companion matrix, the sequence generated from an initial row (u_{-3}, u_{-2}, u_{-1}) is $R(n)^{\mathsf{T}} = (u_{n-3}, u_{n-2}, u_{n-1}) = (u_{-3}, u_{-2}, u_{-1})\, \Phi_C(n)$; extracting the last coordinate with e_3 at index $n+1$ gives $u_n = (u_{-3}, u_{-2}, u_{-1})\, \Phi_C(n+1)\, e_3$. Hence

$$u_n = (u_{-3}, u_{-2}, u_{-1}) \Phi_C(n+1) e_3.$$

Applying this formula to the two initial rows from part c) gives (39).

□

Step 4: the limit.

Theorem 3. $\lim\limits_{n\to\infty} \dfrac{p_n}{q_n} = \gamma.$

Proof. The coboundary $C(n) = U(n)^{-1}\widehat{M}(n)U(n+1)$ telescopes to

$$\Phi_C(n) = U(0)^{-1} \left(\prod_{j=0}^{n-1} (j+1)^4 \right) \Phi_M(n)^{-\mathsf{T}} U(n),$$

whose scalar prefactor cancels in any ratio of two fixed rows. Hence, by Lemma 11,

$$\frac{p_n}{q_n} = \frac{\ell_\gamma^{\mathsf{T}} U(0) \Phi_C(n+1) e_3}{\ell_1^{\mathsf{T}} U(0) \Phi_C(n+1) e_3} = \frac{\ell_\gamma^{\mathsf{T}} \Phi_M(n+1)^{-\mathsf{T}} w_{n+1}}{\ell_1^{\mathsf{T}} \Phi_M(n+1)^{-\mathsf{T}} w_{n+1}}, \qquad w_m = U(m) e_3.$$

By Lemma 10, $S(n)^{\mathsf{T}}$ is the unique recessive solution. Hence the inverse-transpose Miller selection principle (Corollary 1), equivalently the system form of Miller's algorithm in Zahar's theorem [28, Theorem 2.1], selects the recessive initial vector projectively. In the companion coordinates the terminal vector is the fixed vector e_3. Under the coboundary $C(n) = U(n)^{-1}\widehat{M}(n)U(n+1)$, this terminal vector is transported back to the dual M-system as

$$w_m = U(m) e_3.$$

Therefore Corollary 2 gives

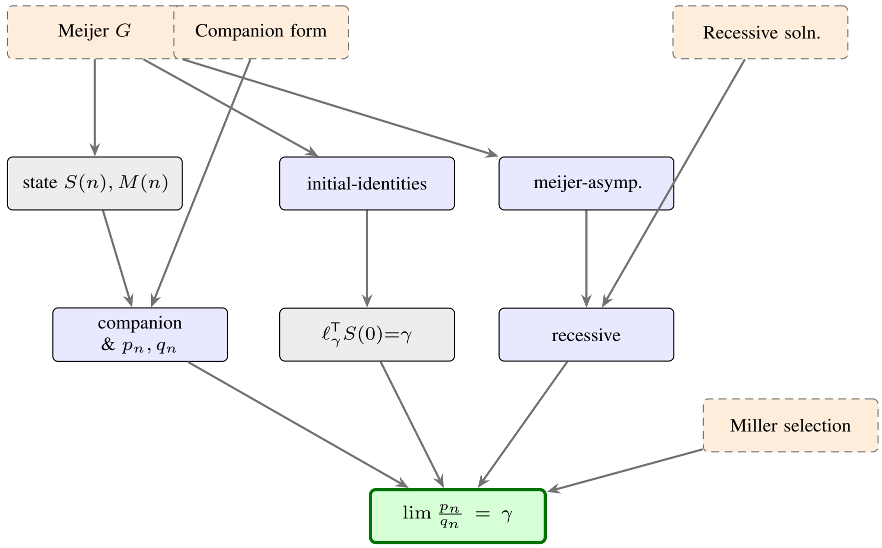
$$\lim_{m\to\infty} \frac{\Phi_M(m)^{-\mathsf{T}} w_m}{\|\Phi_M(m)^{-\mathsf{T}} w_m\|} = \pm \frac{S(0)}{\|S(0)\|}.$$

Combining this with (38) gives

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \frac{\ell_\gamma^{\mathsf{T}} S(0)}{\ell_1^{\mathsf{T}} S(0)} = \frac{\gamma}{1} = \gamma.$$ □

The accompanying notebook contains the entire derivation and computational steps of constructing the formula: retrieving $M(n)$, extracting its asymptotics, computing $\widehat{M}(n)$, converting it to a recurrence via the CVM algorithm, computing the gauged companion-form initial values, and numerically checking the convergence of the recurrence.

5.3 BLUEPRINT DEPENDENCY GRAPH



6 THE SUM $\pi + e$ AS AN APÉRY LIMIT

This section proves Question 3.

6.1 PROOF STRATEGY

The recurrence is a composition of two known polynomial continued fractions, one for π and one for e [17]. Each is written in companion form; their Kronecker product (Section 3.5) gives a fourth-order matrix recurrence for the tensor state vector. A cyclic-vector gauge (Section 3.4) brings this to the companion form appearing in the question. Tracking the numerator and denominator initial vectors and using the coordinate identity $u_n = f_{n+2}g_{n+2}$ yields $p_n/q_n = c_{n+2}/d_{n+2} + x_{n+2}/y_{n+2} \to e + \pi$.

6.2 PROOF

Step 1: the component continued fractions.

We start by selecting two PCFs (Definition 1) from [17] for e and π, rewritten as recurrences of order 2.

The recurrence for e is

$$\frac{1}{e-1} = \frac{1}{1 + \frac{2}{2+\frac{3}{3+\dots}}}, \qquad a_n = n,\ b_n = n,$$

i.e. $f_n = nf_{n-1} + nf_{n-2}$, with two solutions c_n, d_n given by $c_{-1} = 1, c_0 = 1$ and $d_{-1} = 1, d_0 = 0$ satisfying $\lim_{n\to\infty} c_n/d_n = e$. The order-2 recurrence is therefore defined by the matrix

$$M_n^e = \begin{bmatrix} 0 & n \\ 1 & n \end{bmatrix}.$$

The recurrence for π is

$$\frac{4-\pi}{\pi} = \frac{1}{3 + \frac{4}{5+\frac{9}{7+\dots}}}, \qquad a_n = 1+2n,\ b_n = n^2,$$

i.e. $g_n = (1+2n)g_{n-1} + n^2 g_{n-2}$, with two solutions x_n, y_n given by $x_{-1} = 0, x_0 = 4$ and $y_{-1} = 1, y_0 = 1$ satisfying $\lim_{n\to\infty} x_n/y_n = \pi$. The order-2 recurrence is therefore defined by the matrix

$$M_n^\pi = \begin{bmatrix} 0 & n^2 \\ 1 & 2n+1 \end{bmatrix}.$$

Step 2: the Kronecker companion and the question's initial data.

We evaluate M_n as the Kronecker product of the two companion matrices (Section 3.5):

$$M_n = \begin{bmatrix} 0 & n \\ 1 & n \end{bmatrix} \otimes \begin{bmatrix} 0 & n^2 \\ 1 & 2n+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & n^3 \\ 0 & 0 & n & n\,(2n+1) \\ 0 & n^2 & 0 & n^3 \\ 1 & 2n+1 & n & n\,(2n+1) \end{bmatrix}.$$

The Cyclic Vector Method (Section 3.4, [24]) returns the gauge

$$U_n = \begin{bmatrix} 1 & 0 & n^3 & n^3\,(n+1)\,(2n+3) \\ 0 & 0 & n\,(2n+1) & n\,(n+1)\left(5n^2+10n+4\right) \\ 0 & 0 & n^3 & n^2\,(n+1)^2\,(2n+3) \\ 0 & 1 & n\,(2n+1) & (n+1)^2\left(5n^2+10n+4\right) \end{bmatrix},$$

for which $U_n^{-1} M_n U_{n+1}$ is the companion matrix of the recurrence in Question 3:

$$U_n^{-1} M_n U_{n+1} = \begin{bmatrix} 0 & 0 & 0 & \frac{n^3(-n-2)(n+1)^2\left(n^3+n^2-8n-11\right)}{n^3-2n^2-7n-3} \\ 1 & 0 & 0 & \frac{(n+1)^2(n+2)\left(2n^5+3n^4-13n^3-21n^2+4\right)}{n^3-2n^2-7n-3} \\ 0 & 1 & 0 & \frac{(n+2)\left(n^6+9n^5+8n^4-87n^3-249n^2-234n-68\right)}{n^3-2n^2-7n-3} \\ 0 & 0 & 1 & \frac{(n+2)\left(2n^4+n^3-26n^2-48n-19\right)}{n^3-2n^2-7n-3} \end{bmatrix}.$$

The Kronecker product of the initial data is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

whose rows are the tensor state vectors of $(c_n, x_n), (c_n, y_n), (d_n, x_n), (d_n, y_n)$. Because $\frac{c_n}{d_n} + \frac{x_n}{y_n} = \frac{c_n y_n + x_n d_n}{d_n y_n}$, the numerator vector $(1,5,1,1)$ and denominator vector $(1,1,0,0)$ gauge to the initial data of Question 3:

$$\mathbf{p}^\mathsf{T} = (1,5,1,1)\,U_1 = (1,1,20,296), \qquad \mathbf{q}^\mathsf{T} = (1,1,0,0)\,U_1 = (1,0,4,48).$$

Step 3: the tensor coordinate identity.

For any solutions f_n, g_n of the two recurrences, with tensor state vector $\mathbf{v}_n(f,g)^\mathsf{T} = (f_{n-1}g_{n-1}, f_{n-1}g_n, f_n g_{n-1}, f_n g_n)$ and gauged state $\mathbf{w}_n(f,g)^\mathsf{T} = \mathbf{v}_n(f,g)^\mathsf{T} U_{n+1}$, the first coordinate satisfies $u_n = f_{n+2}g_{n+2}$. Consequently the two solutions of Question 3 are

$$p_n = c_{n+2}y_{n+2} + d_{n+2}x_{n+2}, \qquad q_n = d_{n+2}y_{n+2}. \tag{40}$$

Since $U_n^{-1} M_n U_{n+1}$ is companion, $\mathbf{w}_0^\mathsf{T} = (u_{-3}, u_{-2}, u_{-1}, u_0)$ forces $\mathbf{w}_n^\mathsf{T} = (u_{n-3}, u_{n-2}, u_{n-1}, u_n)$. The first column of U_{n+1} is $\mathbf{e}_1$, so $u_{n-3} = \mathbf{w}_n^\mathsf{T}\mathbf{e}_1 = f_{n-1}g_{n-1}$, i.e. $u_n = f_{n+2}g_{n+2}$. Applying this to the numerator and denominator state vectors yields (40).

Step 4: the limit.

We evaluate our limit:

$$\frac{p_n}{q_n} = \frac{c_{n+2}y_{n+2} + d_{n+2}x_{n+2}}{d_{n+2}y_{n+2}} = \frac{c_{n+2}}{d_{n+2}} + \frac{x_{n+2}}{y_{n+2}} \longrightarrow e + \pi.$$

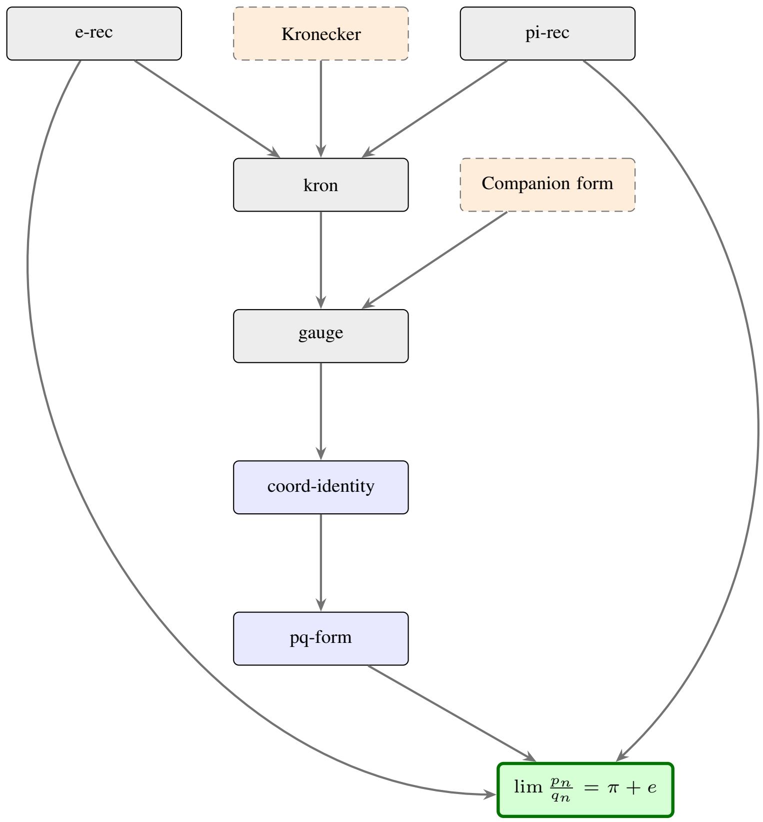
Therefore:

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \pi + e.$$

6.3 Connection to code

The accompanying notebook contains the entire derivation and computational steps of constructing the formula: Starting from M_n^e and M_n^π, computing the Kronecker product matrix $M(n)$, converting it to a recurrence via the CVM algorithm, computing the gauged companion-form initial values manipulating the initial values, and numerically checking the convergence of the recurrence.

6.4 Blueprint dependency graph



7 Harmonic numbers, a polylogarithm and zeta values

This section proves Question 4. We have simplified this sum rigorously while utilizing advanced yet generally applicable summation tools, using the packages `Sigma`, HarmonicSums, and `EvaluateMultiSums` [32, 33, 34, 35]. We will walk through the process here, and we refer the reader to the supplementary Mathematica notebook for the supporting CAS proofs, as well as further details regarding the process itself.

7.1 Proof strategy

The double sum is reduced in two steps.

1. Simplify the double sum into known constants plus two infinite binomial sums, using creative telescoping and difference-field solving (`Sigma`, `EvaluateMultiSums`).
2. Reduce the two remaining binomial sums to known constants, via generating functions and cyclotomic harmonic polylogarithms (`HarmonicSums`), giving the final closed form.

7.2 Simplification to binomial sums

We first simplify the inner sum, then the outer summand. The inner sum is

$$S_m \coloneqq \sum_{k=0}^{m} \binom{m}{k}^2 H_k^2. \tag{41}$$

Lemma 12 (Inner sum)**.** *S_m equals $\binom{2m}{m}$ times a polynomial in sums $\sum_{k=1}^{m} r(k)$ whose summand $r(k)$ does* not *depend on m.*

Proof. Creative telescoping (Section 7.4, [36, 32]) gives a third-order inhomogeneous recurrence for S_m. Difference-field solving [35] produces one particular solution and three linearly independent homogeneous solutions; matching the first three values S_0, S_1, S_2 fixes the linear combination that agrees with S_m for all $m \geq 0$, in the stated form. □

Write $H_i^{(2)} := \sum_{k=1}^{i} \frac{1}{k^2}$ (note $H_i^2 \neq H_i^{(2)}$), abbreviate the two infinite binomial sums

$$B_1 = \sum_{i=1}^{\infty} \frac{1}{i^2 \binom{2i}{i}}, \qquad B_2 = \sum_{i=1}^{\infty} \frac{H_i^{(2)}}{i^2 \binom{2i}{i}}, \tag{42}$$

and substitute Lemma 12 into the outer summand

$$T_m = \frac{S_m}{(m+1)^2 \binom{2m}{m}},$$

which cancels the binomial $\binom{2m}{m}$.

Lemma 13 (Outer reduction)**.**

$$\begin{aligned}\sum_{m=0}^{\infty} T_m = 20\,\mathrm{Li}_4\left(\tfrac{1}{2}\right) + \tfrac{5}{6}\log^4(2) + 10\zeta(2) - \log^2(2)(12 + 5\zeta(2)) \\ - \tfrac{38}{5}\zeta(2)^2 + \tfrac{1}{2}\zeta(3) + \log(2)\left(\tfrac{35}{2}\zeta(3) - 16\right) + 3\zeta(2)B_1 - 3B_2.\end{aligned} \tag{43}$$

Proof. Applying telescoping [32, 35] to $L_A = \sum_{m=0}^{A} T_m$ and taking $A \to \infty$ with `Sigma` and `HarmonicSums` [37, 33] gives (43); the whole step runs automatically within `EvaluateMultiSums` [34]. □

7.3 Evaluation of binomial sums

The two binomial sums (42) are evaluated with `HarmonicSums` [38]. For more details, please consult the supplementary Mathematica notebook.

Lemma 14 (First binomial sum)**.** $B_1 = \dfrac{\zeta(2)}{3}$.

Lemma 15 (Second binomial sum)**.** $B_2 = \dfrac{28\zeta(2)^2}{135}$.

Proof of Lemmas 14 and 15. The process starts by considering the generating functions of these sums obtained by multiplying the summand with x^i. This form converts the recurrence satisfied by the coefficients into a differential equation for the function. This differential equation can be solved in terms of root-valued iterated integrals [39, 38]. These can then be converted into combinations of cyclotomic harmonic polylogarithms [40, 38]. Then, the code uses a table of precomputed closed-form values of these functions, as well as symbolic relations between them, to get the closed forms of the sums [41]. For B_2, the reduction runs

$$\sum_{i=1}^{\infty} \frac{H_i^{(2)}}{i^2 \binom{2i}{i}} \to \sum_{i=1}^{\infty} \frac{x^i H_i^{(2)}}{i^2 \binom{2i}{i}} \to \int_0^x \int \cdots \int \sqrt{\ldots}\, dt_k \ldots dt_1 \to H_{(a_1,b_1),\ldots,(a_k,b_k)}(x) \to \frac{21}{5} H_{(3,0)}(1)^4,$$

This is achieved using relations from the precomputed table. Finally, the evaluation $H_{(3,0)}(1) = \frac{\pi}{3\sqrt{3}}$ is also available in the table. This gives $B_2 = \frac{28\zeta(2)^2}{135}$. A similar computation gives $B_1 = \frac{\zeta(2)}{3}$. □

Theorem 4 (Closed form of the double sum)**.**

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{\binom{m}{k}^2 H_k^2}{(m+1)^2 \binom{2m}{m}} = 20\,\mathrm{Li}_4\left(\tfrac{1}{2}\right) + \tfrac{5}{6}\log^4(2) + 10\zeta(2) - \tfrac{65}{9}\zeta(2)^2 - \log^2(2)(12 + 5\zeta(2)) + \tfrac{1}{2}\zeta(3) + \log(2)\left(\tfrac{35}{2}\zeta(3) - 16\right),$$

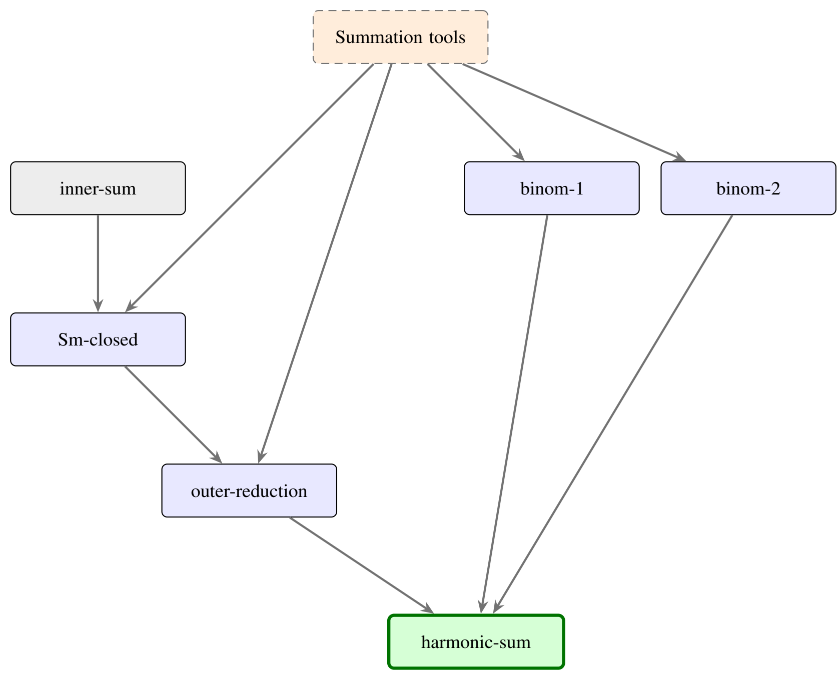
proving Question 4.

Proof. Substitute Lemmas 14 and 15 into (43). The binomial-sum terms give $3\zeta(2)\cdot\frac{\zeta(2)}{3} - 3\cdot\frac{28\zeta(2)^2}{135} = \zeta(2)^2 - \frac{28\zeta(2)^2}{45} = \frac{17}{45}\zeta(2)^2$, which combines with the $-\frac{38}{5}\zeta(2)^2$ term to yield $-\frac{65}{9}\zeta(2)^2$; the other terms are unchanged. $\square$

7.4 Symbolic summation tools

The harmonic-number evaluation is carried out with the RISC packages `Sigma`, `HarmonicSums`, and `EvaluateMultiSums` [32, 33, 34, 35]. The relevant primitives are: creative telescoping [36], which produces linear recurrences for parametrised sums; difference-field solving [35], which expresses the solution through nested sums and products; and the theory of (cyclotomic) harmonic polylogarithms and root-valued iterated integrals [40, 39, 38, 41], which evaluate the remaining binomial sums in closed form.

7.5 Blueprint dependency graph



8 Efficient rational approximation of Catalan's constant G

This section proves Question 5, using the recessive-solution framework of Section 3.9.

8.1 Proof strategy

We take the sequence F_n defined in Section 8.2. The exact relation between its companion recurrence and the matrix $M(n)$ in the question is given by Lemma 16. The positive integral in Lemma 17 proves that

$$\lim_{n\to\infty} F_n^{1/n} = 17 - 12\sqrt{2},$$

and hence, by Lemma 2, that F_n spans the unique recessive solution line.

The initial vector

$$\mathbf{F}_G = (F_0, F_1, F_2)^{\mathsf{T}}$$

is evaluated exactly in Lemma 19, and the nonzero connection condition for all three columns is verified in Lemma 18. Therefore Corollaries 1 and 2 reduce each limit to the two linear forms evaluated in Lemma 19, whose ratio is G.

8.2 The Meijer–G trajectory

We use the Barnes convention of Definition 2. For $n \geq 0$, define

$$\mathcal{F}_n(z) = G^{4,2}_{4,4}\left(z \,\middle|\, \begin{matrix} -1-n, -1-n, n+\frac{5}{2}, n+3 \\ \frac{3}{2}, \frac{1}{2}, 1, 0 \end{matrix}\right), \qquad F_n = \mathcal{F}_n(1),$$

and let

$$Y_n = \big(F_n, (\Theta\mathcal{F}_n)(1), (\Theta^2\mathcal{F}_n)(1)\big), \qquad \Theta = z\frac{d}{dz}.$$

At $z = 1$, the fourth-order terms in the Meijer–G differential equation cancel, and direct expansion gives

$$\Theta^3\mathcal{F}_n(1) = -\frac{(n+2)^3(2n+3)}{7}F_n + \frac{2n^2+8n+5}{14}(\Theta\mathcal{F}_n)(1) + \frac{8n^2+30n+39}{14}(\Theta^2\mathcal{F}_n)(1). \tag{44}$$

so the derivative state has rank three. The Barnes gamma factors give the required contiguous operators: applying the four shifts

$$(-1-n, -1-n, n+\tfrac{5}{2}, n+3) \longmapsto (-2-n, -2-n, n+\tfrac{7}{2}, n+4)$$

and reducing powers of Θ by (44) produces a rational matrix $T_G(n)$ with $Y_{n+1} = Y_n T_G(n)$.

Lemma 16 (Inverse-transpose identification)**.**

$$M(n) = \sigma_G(n)T_G(n)^{-\mathsf{T}}, \qquad \sigma_G(n) = -2(n+2)^2(n+3)^2(2n+5)(2n+7)^2. \tag{45}$$

Consequently, with $P_N = T_G(0)\cdots T_G(N-1)$*,*

$$\mathcal{M}_N = \left(\prod_{m=0}^{N-1}\sigma_G(m)\right)P_N^{-\mathsf{T}}.$$

Proof. We choose the sign of $T_G(n)$ such that the integral representation of Lemma 17 is positive. The ramanujantools trajectory is $-T_G(n)$; the stepwise minus sign is absorbed into the projectively irrelevant scalar $\sigma_G(n)$. The notebook verifies $T_G(n)^{\mathsf{T}}M(n) = \sigma_G(n)I_3$ entry by entry over $\mathbb{Q}(n)$, which is (45); multiplying the single-step identities gives the product form.

Note that the scalar factor affects none of the three ratios (Remark 1). □

8.3 Companion form and the recessive solution

Let $e_1 = (1,0,0)^{\mathsf{T}}$ and define the Krylov matrix

$$U_G(n) = \big[e_1,\ T_G(n)e_1,\ T_G(n)T_G(n+1)e_1\big].$$

We also have

$$C_G(n) = U_G(n)^{-1}T_G(n)U_G(n+1),$$

the companion matrix.

Lemma 17 (The hypergeometric sequence is recessive)**.** *F_n is the unique recessive solution of the companion recurrence.*

Proof.

$$Y_nU_G(n) = (F_n, F_{n+1}, F_{n+2}).$$

The exact determinant is

$$\det U_G(n) = -\frac{15D(n)}{2(n+1)(n+2)^3(n+3)^2(n+4)(2n+3)^2(2n+5)^2(2n+7)},$$

where

$$\begin{aligned} D(n) = {} & 3072n^7 + 61824n^6 + 531184n^5 + 2525522n^4 \\ & + 7175793n^3 + 12183785n^2 + 11446123n + 4589916. \end{aligned}$$

All coefficients of D are positive, so $U_G(n)$ is invertible for every integer $n \geq 0$. Put

$$C_G(n) = U_G(n)^{-1}T_G(n)U_G(n+1).$$

Then

$$T_G(n)U_G(n+1) = U_G(n)C_G(n), \tag{46}$$

and $C_G(n)$ is the row companion matrix of the scalar recurrence satisfied by F_n. Its limit is

$$C_G(n) = C_{G,\infty} + O(n^{-1}), \qquad C_{G,\infty} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -35 \\ 0 & 1 & 35 \end{pmatrix},$$

with characteristic polynomial

$$\chi_G(\lambda) = (\lambda - 1)(\lambda^2 - 34\lambda + 1).$$

The roots are

$$\mu_G = 17 - 12\sqrt{2} = (\sqrt{2} - 1)^4, \qquad 1, \qquad \mu_G^{-1} = 17 + 12\sqrt{2},$$

so μ_G is the unique root of smallest modulus.

We now identify the concrete solution F_n with this recessive mode. The Barnes integral may be transformed, using two beta integrals, two Laplace integrals, and the duplication formula, into

$$\begin{aligned} F_n = \frac{\sqrt{\pi}}{\Gamma(n+1)\Gamma(n+2)} \int_{(0,1)^2 \times (0,\infty)^2} & x^{1/2}(1-x)^n(1-y)^{n+1} \\ & \cdot u^{n+1}v^{n+1} \exp\left(-u - v - 2\sqrt{\frac{uv}{xy}}\right) du\, dv\, dx\, dy. \end{aligned} \tag{47}$$

In particular, $F_n > 0$. After $u = na$ and $v = nb$, the phase is

$$\Phi = \log(1-x) + \log(1-y) + \log a + \log b - a - b - 2\sqrt{\frac{ab}{xy}}.$$

Writing $p = \sqrt{xy}$ and $r = \sqrt{ab}$ gives

$$\Phi + 2 \le 2 \log \frac{p(1-p)}{1+p}.$$

The right-hand side has a unique maximum at $p = t = \sqrt{2} - 1$, and equality throughout forces

$$x = y = t, \qquad a = b = \frac{t}{1+t}.$$

At this point

$$e^{\Phi_* + 2} = t^4 = 17 - 12\sqrt{2}.$$

More explicitly, after $u = na$ and $v = nb$ we have

$$F_n = \frac{\sqrt{\pi}\, n^{2n+4}}{\Gamma(n+1)\Gamma(n+2)} J_n, \qquad J_n = \int_D A(x,y,a,b) e^{n\Phi(x,y,a,b)}\, dx\, dy\, da\, db,$$

with $D = (0,1)^2 \times (0,\infty)^2$ and $A = x^{1/2}(1-y)ab$. The function

$$Ae^{\Phi} \le x^{1/2}(1-x)(1-y)^2 a^2 b^2 e^{-a-b}$$

is integrable on D. Hence $J_n \le e^{(n-1)\Phi_*} \int_D Ae^{\Phi}$, while on every sufficiently small neighborhood of the unique maximizer the functions A and e^{Φ} are bounded below by positive constants arbitrarily close to their values at that point. Therefore $J_n^{1/n} \to e^{\Phi_*}$. Stirling's formula gives

$$\left(\frac{\sqrt{\pi}\, n^{2n+4}}{\Gamma(n+1)\Gamma(n+2)}\right)^{1/n} \longrightarrow e^2,$$

and consequently

$$\lim_{n\to\infty} F_n^{1/n} = e^{\Phi_* + 2} = 17 - 12\sqrt{2} = \mu_G. \tag{48}$$

By Lemma 2, F_n is therefore the unique recessive solution of the companion recurrence. $\square$

Lemma 18 (Convergence)**.**

$$[P_N^{-\mathsf{T}} e_j] \longrightarrow [U_G(0)^{-\mathsf{T}} \mathbf{F}_G], \qquad \mathbf{F}_G = (F_0, F_1, F_2)^{\mathsf{T}}, \qquad j = 1, 2, 3. \tag{49}$$

Proof. Exact asymptotics of the Krylov matrix give

$$\lim_{n\to\infty}\begin{pmatrix}1 & 0 & 0\\ 0 & n & 0\\ 0 & 0 & n^3\end{pmatrix} U_G(n) = \begin{pmatrix}1 & 17 & 577\\ 0 & 24 & 816\\ 0 & -30 & -1020\end{pmatrix}.$$

A right eigenvector of $C_{G,\infty}$ for μ_G is

$$r_G = (17+12\sqrt{2}, -18-12\sqrt{2}, 1)^{\mathsf{T}},$$

and the three limiting pairings

$$96(3-2\sqrt{2}), 96(4-3\sqrt{2}), 120(-4+3\sqrt{2})$$

are all nonzero. Thus the nonzero condition of Corollary 1 holds for every column, and Corollary 2 gives (49). □

8.4 Evaluation of the companion initial vector

Lemma 19 (Companion initial vector and cancellation)**.** *The companion initial vector is*

$$\mathbf{F}_G = \sqrt{\pi}\begin{pmatrix} 150G - 128\log 2 - \dfrac{146}{3} \\ \dfrac{24745}{4}G - \dfrac{14624}{3}\log 2 - \dfrac{823511}{360} \\ \dfrac{7225281}{32}G - \dfrac{886784}{5}\log 2 - \dfrac{2818419551}{33600}\end{pmatrix}, \tag{50}$$

and, with $\widetilde{A}_G = A\,U_G(0)^{-\mathsf{T}}$,

$$\widetilde{A}_G\mathbf{F}_G = 450\sqrt{\pi}\begin{pmatrix}G\\ 1\end{pmatrix}. \tag{51}$$

Proof. The initial coboundary matrix is

$$U_G(0) = \begin{pmatrix}1 & \dfrac{152}{5} & \dfrac{195477}{175}\\ 0 & \dfrac{1723}{90} & \dfrac{1963751}{2800}\\ 0 & -\dfrac{143}{18} & -\dfrac{165201}{560}\end{pmatrix}.$$

The Barnes-moment calculation at $n=0$, using the three moments

$$J_1 = \sqrt{\pi}\left(16\log 2 - \frac{167}{15}\right), \qquad J_2 = \sqrt{\pi}\left(48\log 2 - \frac{333}{10}\right), \qquad J_{3/2} = \sqrt{\pi}\left(\frac{75}{2}G - \frac{2063}{60}\right),$$

together with the partial-fraction reduction of the Barnes kernel, and $\mathbf{F}_G = U_G(0)^{\mathsf{T}}Y_0^{\mathsf{T}}$, give (50). With

$$\widetilde{A}_G = A\,U_G(0)^{-\mathsf{T}} = \frac{1}{382493}\begin{pmatrix}30102216645 & -19896711516 & 525137760\\ 32863650750 & -21712357800 & 573048000\end{pmatrix},$$

substitution of (50) produces the exact cancellation (51). □

Theorem 5 (Catalan limit)**.** *For each* $j=1,2,3$, $\lim_{N\to\infty}\dfrac{P_{N,j}}{Q_{N,j}} = G$.

Proof. By Lemmas 18 and 19, for every $j=1,2,3$,

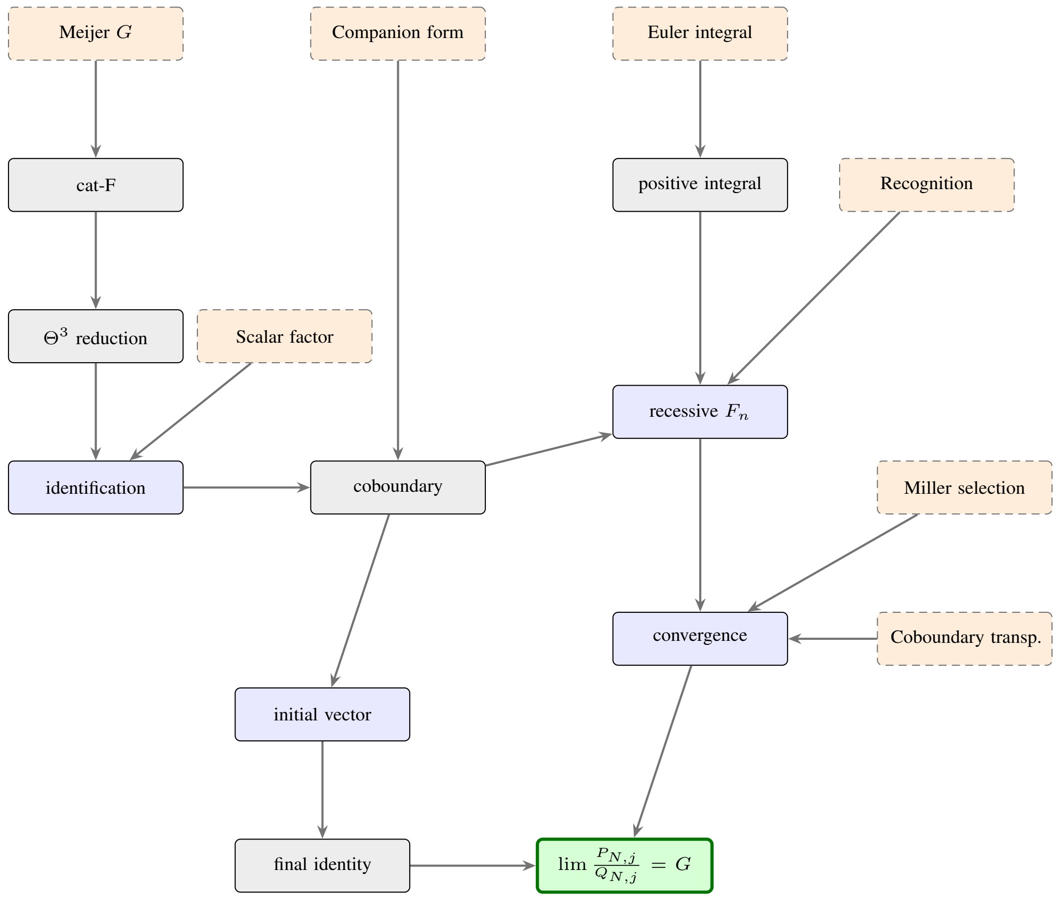
$$A\mathcal{M}_N e_j = \gamma_{N,j}\left(450\sqrt{\pi}\begin{pmatrix}G\\ 1\end{pmatrix} + o(1)\right), \qquad \gamma_{N,j}\neq 0 \text{ eventually.}$$

The second component is nonzero for all sufficiently large N, so $P_{N,j}/Q_{N,j}\to G$. □

8.5 CONNECTION WITH CODE

The notebook constructs the Meijer–G trajectory in ramanujantools notation and records a SymPy-only exact certificate. It verifies (45), the determinant and coboundary (46), the limiting characteristic polynomial, the saddle equations and value, the three nonzero column connections, the initial vector (50), and the final cancellation (51), all over $\mathbb{Q}(n)$ or exact algebraic extensions; numerical products are a sanity check.

8.6 BLUEPRINT DEPENDENCY GRAPH



9 A SERIES FOR $\zeta(2) + \zeta(3)$

This section proves Question 6, using the framework of Section 3.9.

9.1 PROOF STRATEGY

We first reduce the series in the question to a matrix limit. The auxiliary quotient L_N of Lemma 20 satisfies

$$L_3 = \frac{2077}{720}, \qquad u_n = L_{n+3} - L_{n+2},$$

so it is enough to prove that $L_N \to \zeta(2) + \zeta(3)$.

The companion recurrence and its exact coboundary with the auxiliary matrix walk are given in Lemma 21. We then take the sequence f_n defined in (55). The factorization and tail estimates in Lemma 22 prove that f_n satisfies the companion recurrence and that $f_n = O(n^{-2})$. The asymptotic basis in Lemma 23 then shows that f_n spans the unique recessive solution line.

Finally, Lemma 24 evaluates

$$\mathbf{f}_0 = (f_0, f_1, f_2)^\mathsf{T}$$

exactly and computes its image under $K(0)^{-\mathsf{T}}$. The required terminal connection coefficient is $N+1$, as used in Theorem 6. The inverse-transpose selection and coboundary transport therefore give the limit of L_N, and Lemma 20 converts this matrix limit back into the series in the question.

9.2 The auxiliary quotient

For $n \geq 0$, put

$$M(n) = \begin{pmatrix} \dfrac{n+2}{n+1} & -\dfrac{(n+2)(3n+4)}{(n+1)(2n+3)} & \dfrac{3n^3+21n^2+43n+27}{(2n+2)(2n+3)} \\ \dfrac{3n+6}{(n+1)^2} & -\dfrac{(n+2)(n^2+11n+13)}{(n+1)^2(2n+3)} & \dfrac{3n^4+27n^3+102n^2+165n+93}{(n+1)^2(4n+6)} \\ \dfrac{3n+6}{(n+1)^3} & -\dfrac{3(n+2)(n^2+5n+5)}{(n+1)^3(2n+3)} & \dfrac{7n^4+54n^3+165n^2+226n+114}{(n+1)^3(4n+6)} \end{pmatrix}.$$

Let $\Phi_M(N) = M(0)M(1)\cdots M(N-1)$, $p = (0,1,-2)$, $q = (-1,1,-1)$, and $e_1 = (1,0,0)^{\mathsf{T}}$. Define

$$L_N = \frac{p\Phi_M(N)e_1}{q\Phi_M(N)e_1}.$$

Lemma 20 (Bridge to the question)**.** *a) We get*

$$q\Phi_M(N)e_1 = -(N+1), \qquad L_3 = \frac{2077}{720}, \qquad u_n = L_{n+3} - L_{n+2}, \tag{52}$$

b) The last expression has the stated initial values and satisfies the recurrence of Question 6.

Proof. Please refer to the accompanying notebook. ☐

9.3 The inverse-transpose companion

Lemma 21 (Inverse-transpose companion)**.** *Let $D(n) = M(n)^{-\mathsf{T}}$, let $C(n)$ be its companion form, and let $K(n)$ be the companion coboundary matrix (Section 3.4). Then*

$$D(n)K(n+1) = K(n)C(n), \qquad \det K(n) = \frac{36n^4+297n^3+897n^2+1169n+553}{(n+2)^3(n+3)}, \tag{53}$$

so $K(n)$ is invertible for every integer $n \geq 0$, and

$$K(0) = \begin{pmatrix} 1 & -\dfrac{29}{2} & -\dfrac{869}{12} \\ 0 & \dfrac{17}{2} & \dfrac{419}{12} \\ 0 & -\dfrac{7}{2} & -\dfrac{35}{3} \end{pmatrix}. \tag{54}$$

Proof. The exact rational entries of $C(n)$ and $K(n)$, the coboundary (53), the determinant, and $K(0)$ are recorded and verified in the notebook. ☐

9.4 A solution tending to zero

Put

$$P(n) = 36n^4+297n^3+897n^2+1169n+553,$$

$$h_n = \frac{P(n)}{(n+1)^2(n+2)^3(n+3)(3n+5)}, \qquad R_n = \frac{(2n+1)(3n+5)}{5(n+1)^2}\binom{2n}{n},$$

and define

$$g_n = -R_n \sum_{k=n}^{\infty} \frac{h_k}{R_{k+1}}, \qquad f_n = -(n+1)^2 \sum_{j=n}^{\infty} \frac{g_j}{(j+2)^2}. \tag{55}$$

Define also

$$r_0(n) = \frac{(n+2)^2}{(n+1)^2}, \qquad r_1(n) = \frac{2(n+1)(2n+3)(3n+8)}{(n+2)^2(3n+5)}, \qquad r_2(n) = \frac{(n+1)^2(n+2)(3n+5)P(n+1)}{(n+3)^2(n+4)(3n+8)P(n)}.$$

Direct cancellation gives $R_{n+1}/R_n = r_1(n)$ and $h_{n+1}/h_n = r_2(n)$. Subtracting consecutive tails in (55) gives

$$g_{n+1} - r_1(n)g_n = h_n, \qquad f_{n+1} - r_0(n)f_n = g_n.$$

The notebook verifies the exact Ore factorization

$$\mathcal{L}_C = \big(S - r_2(n)\big)\big(S - r_1(n)\big)\big(S - r_0(n)\big), \qquad S\,a(n) = a(n+1)S,$$

where $\mathcal{L}_C$ is the scalar recurrence operator encoded by $C(n)$; hence f_n satisfies the companion recurrence exactly.

Lemma 22 (A solution tending to zero)**.** *The sequence f_n of* (55) *satisfies the companion recurrence of Lemma 21 and $f_n = O(n^{-2}) \to 0$.*

Proof. The Ore factorization above shows f_n solves the companion recurrence. Stirling's formula gives

$$h_n = O(n^{-3})$$

and

$$R_n \asymp 4^n n^{-1/2}.$$

Hence

$$h_n/R_{n+1} = O(4^{-n} n^{-5/2}),$$

so the inner tail has the same order and

$$g_n = O(n^{-3}).$$

Therefore

$$f_n = O\left(n^2 \sum_{j=n}^{\infty} j^{-5}\right) = O(n^{-2}) \to 0.$$

□

Lemma 23 (Recessiveness)**.** *f_n is the unique recessive solution (Definition 4) of the companion recurrence.*

Proof. For the exact companion $C(n)$, `LinearRecurrence(C).asymptotics()` returns

$$\left[\frac{4^n}{\sqrt{n}},\ n^2,\ n^{-2}\right], \tag{56}$$

a fundamental asymptotic basis whose first two classes diverge and whose third tends to zero. By Lemma 22, f_n is a nonzero solution with $f_n \to 0$, it spans the unique recessive line. □

9.5 Initial values and projective limit

Lemma 24 (Initial values)**.** *We have*

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 - \zeta(2) + \zeta(3) \\ 6\zeta(2) + 4\zeta(3) - \dfrac{29}{2} \\ \dfrac{75}{2}\zeta(2) + 9\zeta(3) - \dfrac{869}{12} \end{pmatrix}. \tag{57}$$

Furthermore, setting

$$w := K(0)^{-\mathsf{T}} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}, \tag{58}$$

we get

$$[\Phi_M(N)e_1] \longrightarrow [w], \qquad w = K(0)^{-\mathsf{T}} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}.$$

Proof. The values (57) are a direct Mathematica evaluation of the defining tails; they are reproduced in the notebook. multiplying by $K(0)^{-\mathsf{T}}$ from (54) and pairing with $p = (0, 1, -2)$ and $q = (-1, 1, -1)$ gives (58).

For the second part: the proof of the inverse-transpose selection principle in Section 3.9 uses only a fundamental asymptotic basis with one uniquely recessive scale. It therefore applies to (56); equivalently, this is the standard Miller backward-recurrence principle. The notebook verifies that the moving terminal vector $K(N)^{\mathsf{T}} e_1$ has recessive connection coefficient exactly $N + 1$, hence never lies in the complementary dual plane. The claim then follows from (53). ☐

Theorem 6 (Series for $\zeta(2) + \zeta(3)$)**.**

$$\frac{2077}{720} + \sum_{n=1}^{\infty} u_n = \zeta(2) + \zeta(3).$$

Proof. Using (54) and (57), one obtains

$$w = \begin{pmatrix} 1 - \zeta(2) + \zeta(3) \\ -\zeta(2) + 3\zeta(3) \\ 2\zeta(3) \end{pmatrix}.$$

Thus

$$p\,w = -\big(\zeta(2) + \zeta(3)\big), \qquad q\,w = -1,$$

and therefore

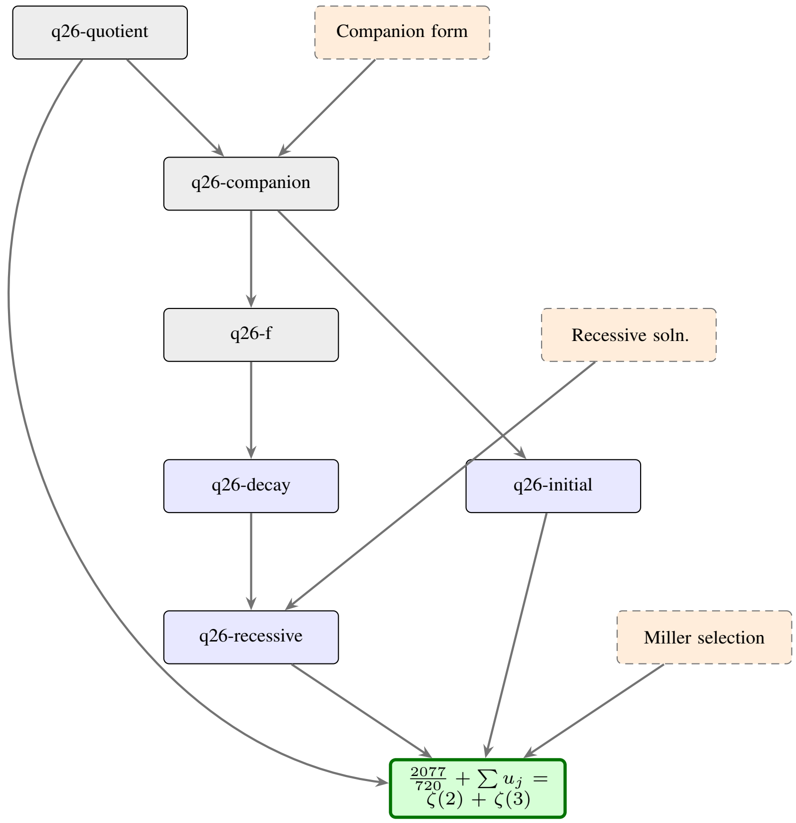
$$L_N = \frac{p\Phi_M(N)e_1}{q\Phi_M(N)e_1} \longrightarrow \frac{p\,w}{q\,w} = \zeta(2) + \zeta(3).$$

By Lemma 20, $\frac{2077}{720} + \sum_{n=1}^{N} u_n = L_{N+3}$; letting $N \to \infty$ completes the proof. ☐

9.6 Connection to code

The notebook verifies the inverse-transpose companion and coboundary, the exact Ore factorization, the two tail-difference identities, `LinearRecurrence(C).asymptotics()`, the decay $f_n = O(n^{-2})$, the Mathematica values of f_0, f_1, f_2, the nonzero terminal connection, the multiplication by $K(0)^{-\mathsf{T}}$, and the finite bridge to the sequence in the question.

9.7 BLUEPRINT DEPENDENCY GRAPH



10 EFFICIENT FOUR-TERM RECURRENCE FOR $\zeta(2) + \zeta(3)$

This section proves Question 7, using the framework of Section 3.9.

10.1 PROOF STRATEGY

The recurrence in the question is already a matrix walk, with step matrix $R(n)$ defined in (59). We take the sequence F_n defined in Section 10.2. Its companion recurrence is established in Lemma 26, while the integral and asymptotic arguments in Lemmas 27 and 28 prove that F_n spans the unique recessive solution line.

The exact rational coboundaries relating the inverse-transpose companion walk to $R(n)$ are given in Lemma 25. At the regular initial index, the vector

$$\mathbf{F}_\zeta = (F_2, F_3, F_4)^\mathsf{T}$$

and the two required linear forms on it are evaluated. The invariant-pairing principle (Corollary 3) therefore identifies the limiting ratio p_n/q_n with the ratio of these two pairings, namely $\zeta(2) + \zeta(3)$.

10.2 THE TARGET RECURRENCE AND THE HYPERGEOMETRIC TRAJECTORY

Let

$$R(n) = \begin{pmatrix} 0 & 0 & D_n/A_n \\ 1 & 0 & -C_{n+1}/A_{n+1} \\ 0 & 1 & B_{n+2}/A_{n+2} \end{pmatrix}. \tag{59}$$

Then every solution u of the recurrence in the question satisfies $(u_n, u_{n+1}, u_{n+2})R(n) = (u_{n+1}, u_{n+2}, u_{n+3})$. Its limiting characteristic polynomial is

$$Q(\rho) = \rho^3 - \frac{55}{64}\rho^2 + \frac{1}{2048}\rho - \frac{1}{1048576}.$$

Consider

$$\mathcal{F}_n(z) = {}_4F_3\left(\begin{matrix} n+1, n+1, n+1, n+1 \\ 2n+1, 2n+1, 2n+2 \end{matrix}; z\right), \qquad F_n = \mathcal{F}_n(1).$$

At $z = 1$ the leading terms of the fourth-order hypergeometric differential equation cancel, so the derivative state

$$Y_n = \big(F_n, (\Theta\mathcal{F}_n)(1), (\Theta^2\mathcal{F}_n)(1)\big), \qquad \Theta = z\frac{d}{dz},$$

has rank three. The upper- and lower-parameter contiguous identities produce an exact rational trajectory matrix $T(n)$ with

$$Y_{n+1} = Y_n T(n). \tag{60}$$

In ramanujantools notation, this is the trajectory with initial point $(1, 1, 1, 1; 1, 1, 2)$ and direction $(1, 1, 1, 1; 2, 2, 2)$ at $z = 1$.

Lemma 25 (Dual companion and rational coboundary)**.** *Put $M(n) = T(n)^{-\mathsf{T}}$, $e_1 = (1, 0, 0)^{\mathsf{T}}$, and the dual Krylov matrix $K(n) = \big[e_1, M(n)e_1, M(n)M(n+1)e_1\big]$. Then*

$$M(n)K(n+1) = K(n)C(n), \tag{61}$$

with $C(n)$ a row companion matrix, and there exists $U(n) \in \mathrm{GL}_3(\mathbb{Q}(n))$ with

$$C(n)U(n+1) = U(n)R(n), \tag{62}$$

so the recurrence of Question 7 is a rational coboundary of the dual companion. Both $K(n)$ and $U(n)$ are invertible for every integer $n \geq 2$.

Proof. The determinant certificate

$$\det K(n) = \frac{n^2(n+1)^4(n+2)P_8(n)}{2048(n-1)(2n-1)(2n+1)^5(2n+3)^4(2n+5)},$$

$$P_8(n) = 946n^8 + 10664n^7 + 50850n^6 + 134048n^5 + 214320n^4 + 214163n^3 + 131918n^2 + 46311n + 7128,$$

shows $K(n)$ is invertible for $n \geq 2$; the symbolic certificate provides the nine entries of $U(n)$ (retained in the notebook), with

$$\det U(n) = -\frac{n^5(n+1)^6(n+2)^9(n+3)^6}{2(2n+1)^2(2n+3)^6(2n+5)^5(2n+7)^3(946n^2 + 4515n + 5399)P_8(n)},$$

so $U(n)$ is invertible for $n \geq 2$, and

$$U(n) \longrightarrow \begin{pmatrix} -19/1301696 & -2221/166617088 & -122075/10663493632 \\ 160/20339 & 8865/1301696 & 974379/166617088 \\ -284288/20339 & -244150/20339 & -13419385/1301696 \end{pmatrix}$$

with determinant $-1/117298429952$; thus the coboundary has no singularity at infinity and preserves the isolated dual mode. □

10.3 The unique recessive primal solution

Lemma 26 (Primal companion)**.** *On the primal side, $V(n) = \big[e_1, T(n)e_1, T(n)T(n+1)e_1\big]$ satisfies $T(n)V(n+1) = V(n)C_F(n)$ and $Y_nV(n) = (F_n, F_{n+1}, F_{n+2})$, is invertible for $n \geq 2$, and has limiting companion*

$$C_{F,\infty} = \begin{pmatrix} 0 & 0 & 1048576 \\ 1 & 0 & -901120 \\ 0 & 1 & 512 \end{pmatrix},$$

with characteristic polynomial

$$P(\lambda) = \lambda^3 - 512\lambda^2 + 901120\lambda - 1048576. \tag{63}$$

Proof. The exact determinant

$$\det V(n) = \frac{1024(n-1)(2n-1)(2n+1)^4(2n+3)^4(2n+5)P_7(n)}{n^8(n+1)^9(n+2)^4},$$

$$P_7(n) = 946n^7 + 6235n^6 + 16523n^5 + 22575n^4 + 17131n^3 + 7350n^2 + 1708n + 176,$$

is nonzero for $n \geq 2$; expansion gives $C_{F,\infty}$ and (63). □

Lemma 27 (Euler integral and root rate). *For $n \geq 1$,*

$$F_n = \frac{\Gamma(2n+1)^2\Gamma(2n+2)}{\Gamma(n+1)^4\Gamma(n)^2}\int_{[0,1]^3}\frac{(t_1t_2t_3)^n(1-t_1)^{n-1}(1-t_2)^{n-1}(1-t_3)^n}{(1-t_1t_2t_3)^{n+1}}\,dt_1dt_2dt_3, \tag{64}$$

and $\lim_{n\to\infty} F_n^{1/n} = \mu$ where $\mu = 64\frac{\tau^3(1-\tau)^3}{1-\tau^3}$ and τ is the root of $\tau^3+\tau^2+\tau-1=0$ in $(0,1)$; moreover $P(\mu)=0$.

Proof. Three applications of Euler's beta integral (Lemma 1) give (64). Set

$$g(t_1,t_2,t_3) = \frac{t_1t_2t_3(1-t_1)(1-t_2)(1-t_3)}{1-t_1t_2t_3}.$$

It extends continuously by zero on the boundary. With $r = t_1t_2t_3$, the critical equations

$$\frac{1}{t_i} - \frac{1}{1-t_i} + \frac{r}{t_i(1-r)} = 0$$

reduce to $1 = t_i(2-r)$, so the unique maximizer is (τ,τ,τ) with

$$\tau^3+\tau^2+\tau-1 = 0. \tag{65}$$

Writing $I_n = \int_{[0,1]^3} g^n a$, its amplitude is

$$a = \frac{1}{(1-t_1)(1-t_2)(1-t_1t_2t_3)}.$$

A direct simplification gives

$$g^2a = \frac{(t_1t_2t_3)^2(1-t_1)(1-t_2)(1-t_3)^2}{(1-t_1t_2t_3)^3} \leq 1$$

since $1-t_i \leq 1-t_1t_2t_3$. Thus $I_n \leq g(\tau,\tau,\tau)^{n-2}$ for $n \geq 2$. Conversely, on a small compact neighborhood of the unique maximizer, a has a positive lower bound and $g \geq g(\tau,\tau,\tau)-\varepsilon$. It follows that

$$I_n^{1/n} \to g(\tau,\tau,\tau).$$

Stirling's formula gives a root rate 64 for the gamma prefactor, hence

$$\lim_{n\to\infty} F_n^{1/n} = \mu, \qquad \mu = 64\frac{\tau^3(1-\tau)^3}{1-\tau^3}. \tag{66}$$

Finally $\operatorname{Res}_\tau\big(\tau^3+\tau^2+\tau-1,(1-\tau^3)\lambda-64\tau^3(1-\tau)^3\big) = 2P(\lambda)$, so $P(\mu)=0$. ☐

Lemma 28 (Recessiveness). *F_n is the unique recessive solution of the primal companion recurrence.*

Proof.

$$\operatorname{disc} P = -2705919006677663744 < 0$$

and $P(1) < 0 < P(2)$, so P has one real root $\mu \in (1,2)$ and one complex-conjugate pair; since the product of the three roots is 2^{20}, each nonreal root has modulus $\sqrt{2^{20}/\mu} > \mu$. As $F_n^{1/n} \to \mu$, Lemma 2 makes F_n the unique recessive solution. ☐

10.4 The companion initial vector and the two invariant pairings

We work at the regular point $n=2$. Exact evaluation of the rational hypergeometric summands gives

$$\mathbf{F}_\zeta = \begin{pmatrix}F_2\\F_3\\F_4\end{pmatrix} = \begin{pmatrix}111240-51840\zeta(2)-21600\zeta(3)\\ \dfrac{344246000}{3}-76020000\zeta(2)+8568000\zeta(3)\\ -\dfrac{104516730500}{3}+3457440000\zeta(2)+24251472000\zeta(3)\end{pmatrix}. \tag{67}$$

For example, the summand of F_2 is

$$\frac{4320(k+1)(k+2)}{(k+3)^3(k+4)^3(k+5)},$$

and partial fractions reduce its sum to $1, \zeta(2), \zeta(3)$. The same calculation at $n = 3, 4$ gives the other two entries. Equivalently,

$$\mathbf{F}_\zeta = V(2)^\mathsf{T} Y_2^\mathsf{T},$$

where

$$V(2) = \begin{pmatrix} 1 & 16975/6 & 26659675/24 \\ 0 & -78575/162 & -39985225/648 \\ 0 & -24500/81 & -46416475/324 \end{pmatrix}.$$

Let

$$\mathbf{p}_0 = (p_0, p_1, p_2), \qquad \mathbf{q}_0 = (q_0, q_1, q_2).$$

Shift these rows to $n = 2$ by multiplying by $R(0)R(1)$, and transport them through the two rational coboundaries. After multiplying both rows by the same harmless rational scalar, the resulting matrix in the primal companion basis is

$$\widetilde{A}_\zeta = \begin{pmatrix} 9523287821075 & -6011887077 & 10606374 \\ 3345036156000 & -2111635008 & 3725352 \end{pmatrix}. \tag{68}$$

More precisely, the notebook verifies

$$\frac{16197}{3160208}\widetilde{A}_\zeta = \begin{pmatrix} \mathbf{p}_0 \\ \mathbf{q}_0 \end{pmatrix} R(0)R(1)U(2)^{-1}K(2)^{-1}V(2)^{-\mathsf{T}}.$$

Substitution of (67) gives the exact identity

$$\widetilde{A}_\zeta \mathbf{F}_\zeta = 7316671572000 \begin{pmatrix} \zeta(2) + \zeta(3) \\ 1 \end{pmatrix}. \tag{69}$$

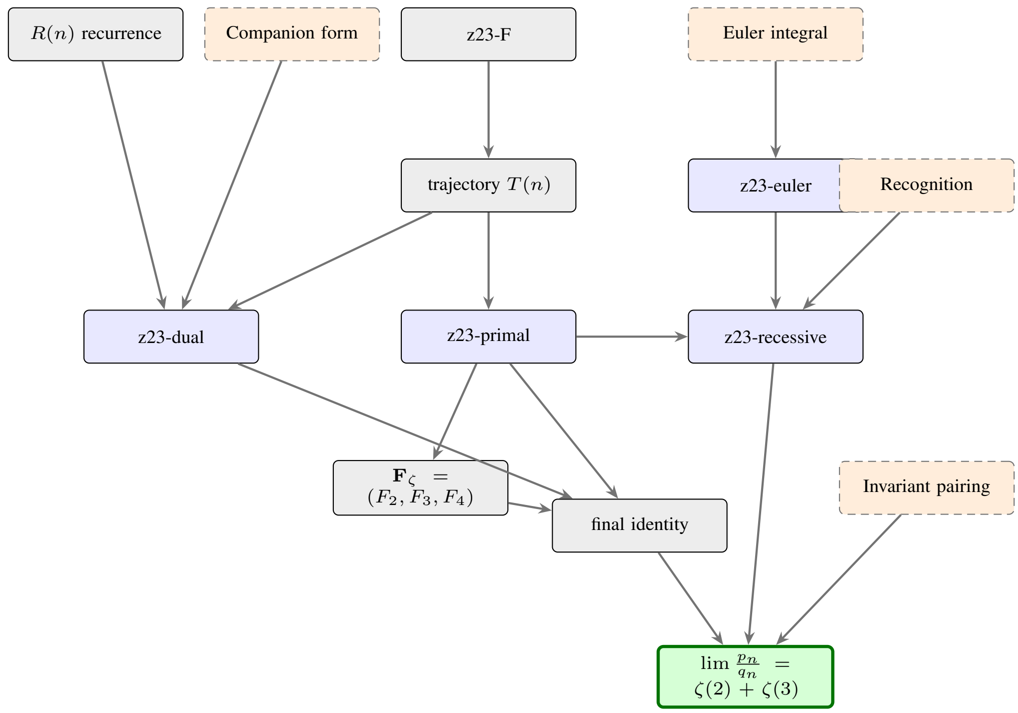
Theorem 7 (Apéry limit for $\zeta(2) + \zeta(3)$)**.** $\lim_{n\to\infty} \frac{p_n}{q_n} = \zeta(2) + \zeta(3)$.

Proof. The second component of (69) is nonzero. By Lemma 28 and the invariant-pairing principle (Corollary 3), the two target solutions share the dominant dual mode, with coefficients equal, up to a common normalization, to the two invariant pairings of (69). Hence $p_n/q_n \to \zeta(2) + \zeta(3)$, and $q_n \neq 0$ for all sufficiently large n. $\square$

10.5 Connection to code

The notebook reconstructs the ${}_4F_3$ trajectory, forms its inverse transpose, constructs both Krylov companion matrices, and verifies (62) over $\mathbb{Q}(n)$, together with the determinant certificates, the limiting polynomial (63), the resultant and discriminant, the three partial-fraction evaluations (67), the transported matrix (68), and the final cancellation (69).

10.6 Blueprint dependency graph



11 Very fast rational approximation of $\sqrt{10005}/\pi$

This section addresses Question 8, using the framework of Section 3.9.

11.1 Proof strategy

We take the sequence f_n defined in Section 11.2. The exact relation between its companion recurrence and the matrix $M(n)$ in the question is given by Lemmas 29 and 30. The integral estimate in Lemma 31 determines the growth of f_n, and Lemmas 32 and 33 show that this growth corresponds to the unique recessive solution line.

The nonzero connection condition for all four columns is verified in Lemma 34. It remains to evaluate the initial data needed for the final quotient. Normalized tails of the Chudnovsky series evaluate exactly the two required linear forms on

$$\mathbf{f}_{\pi,0} = (f_0, f_1, f_2, f_3)^{\mathsf{T}},$$

and their ratio is $\sqrt{10005}/\pi$. The inverse-transpose selection and coboundary transport principles (Corollaries 1 and 2) then prove the claimed limit for all four columns.

11.2 The trajectory and the matrix in the question

Set $q = 53360^{-3}$, $z_0 = -q$, and define

$$\{_n(z) = {}_4F_3\left(\begin{matrix} n+1, n+\frac{7}{6}, n+\frac{3}{2}, n+\frac{11}{6} \\ 2n+2, 2n+2, 2n+2 \end{matrix}; z\right), \qquad f_n = \{_n(z_0). \tag{70}$$

This is the ${}_4F_3$ trajectory with initial point $\left(1, \frac{7}{6}, \frac{3}{2}, \frac{11}{6}; 2, 2, 2\right)$ and direction

$$(1, 1, 1, 1; 2, 2, 2),$$

i.e. the word `1111222`. Let

$$Y_n = \left(f_n, (\Theta\{_n)(z_0), (\Theta^2\{_n)(z_0), (\Theta^3\{_n)(z_0)\right).$$

The standard contiguous relations produce a rational matrix $T_\pi(n)$ with

$$Y_{n+1} = Y_n T_\pi(n). \tag{71}$$

Lemma 29 (Inverse-transpose identification). *Let $M(n)$ be the matrix in Question 8. Then*

$$M(n) = \sigma_\pi(n)T_\pi(n)^{-\mathsf{T}}, \qquad \sigma_\pi(n) = \frac{87512470880256000\,(n+1)^2(2n+3)^2}{(6n+7)(6n+11)}. \tag{72}$$

Proof. Exact cancellation of all sixteen entries gives (72); the scalar $\sigma_\pi(n)$ is projectively irrelevant by Remark 1. □

Thus, the challenge product is, projectively, the inverse transpose of the primal trajectory product (Remark 1).

Lemma 30 (Companion coboundary and characteristic polynomial). *Let $e_1 = (1,0,0,0)^{\mathsf{T}}$ and*

$$K_\pi(n) = \big[e_1, T_\pi(n)e_1, T_\pi(n)T_\pi(n+1)e_1, T_\pi(n)T_\pi(n+1)T_\pi(n+2)e_1\big].$$

Then

$$T_\pi(n)K_\pi(n+1) = K_\pi(n)C_\pi(n), \qquad Y_nK_\pi(n) = (f_n, f_{n+1}, f_{n+2}, f_{n+3}), \tag{73}$$

with $C_\pi(n)$ a fourth-order row companion matrix, invertible for every $n \geq 0$, and $C_\pi(n) = C_{\pi,\infty} + O(n^{-1})$ with characteristic polynomial

$$\begin{aligned}\chi_\pi(\lambda) = \lambda^4 - \frac{256(q+3)}{q^2}\lambda^3 + \frac{8192(3q^2-83q+24)}{q^4}\lambda^2 \\ - \frac{2^{20}(q^3+20q^2+34q+16)}{q^6}\lambda + \frac{2^{24}(q+1)^2}{q^6},\end{aligned} \tag{74}$$

and $\gcd(\chi_\pi, \chi_\pi') = 1$.

Proof. The exact determinant of $K_\pi(n)$ has numerator of one sign and denominator of the opposite fixed sign for $n \geq 0$, so $K_\pi(n)$ is invertible. The notebook checks every coefficient. With

$$D_n = \operatorname{diag}(1, n, n^2, n^3),$$

there is a rational matrix L_π with

$$D_nT_\pi(n)D_{n+1}^{-1} = L_\pi + O(n^{-1}).$$

Define

$$B_\pi = [e_1, L_\pi e_1, L_\pi^2 e_1, L_\pi^3 e_1].$$

We get

$$D_nK_\pi(n) = B_\pi + O(n^{-1})$$

and

$$C_{\pi,\infty} = B_\pi^{-1}L_\pi B_\pi,$$

whose characteristic polynomial is (74). The notebook verifies this identity and $\gcd(\chi_\pi, \chi_\pi') = 1$ exactly. □

11.3 The recessive hypergeometric solution

Lemma 31 (Euler integral and Laplace rate). *For $a_1 = \frac{7}{6}$, $a_2 = \frac{3}{2}$, $a_3 = \frac{11}{6}$, $b_1 = \frac{5}{6}$, $b_2 = \frac{1}{2}$, $b_3 = \frac{1}{6}$,*

a)

$$f_n = P(n)\int_{[0,1]^3}\prod_{j=1}^{3} t_j^{n+a_j-1}(1-t_j)^{n+b_j-1}(1+qt_1t_2t_3)^{-(n+1)}\,dt_1dt_2dt_3, \tag{75}$$

where

$$P(n) = \prod_{j=1}^{3}\frac{\Gamma(2n+2)}{\Gamma(n+a_j)\Gamma(n+b_j)}$$

b)
$$f_n = c_\pi \mu_\pi^n(1+O(n^{-1})), \qquad c_\pi > 0, \qquad \mu_\pi = 64\, t_*^4(1-t_*)^2, \tag{76}$$
where $t_* \in (0, 1/2)$ *is the unique root of* $1 - 2t - qt^4 = 0$

c) In particular,
$$f_{n+j}/f_n \to \mu_\pi^j$$
for $j = 0, 1, 2, 3$.

Proof. The phase is
$$\Phi(t_1,t_2,t_3) = \sum_{j=1}^{3} \log t_j + \sum_{j=1}^{3} \log(1-t_j) - \log(1+qt_1t_2t_3).$$
The Hessian of the first two sums is diagonal, with diagonal entries
$$-\frac{1}{t_j^2} - \frac{1}{(1-t_j)^2} \le -2.$$
For the perturbation $-\log(1+qt_1t_2t_3)$, the diagonal Hessian entries have absolute value at most q^2 and the off-diagonal entries have absolute value at most q. Gershgorin's estimate therefore gives
$$v^\mathsf{T}\nabla^2\Phi(t)v \le (-2+q^2+2q)\|v\|^2 < 0.$$
Thus Φ is strictly concave. It tends to $-\infty$ at the boundary, so it has a unique maximizer; symmetry forces it to be (t_*, t_*, t_*). The critical equation is
$$\frac{1}{t} - \frac{1}{1-t} - \frac{qt^2}{1+qt^3} = 0,$$
which is equivalent to
$$1 - 2t - qt^4 = 0. \tag{77}$$
The left-hand side is strictly decreasing and changes sign between 0 and $1/2$, so this root is unique.

All exponents in the amplitude of (75) are greater than -1, hence the amplitude is integrable on the cube. On a fixed neighborhood of the nondegenerate maximizer, the usual quadratic Taylor expansion and the substitution $u = \sqrt{n}(t - t_*)$ give the three-dimensional Laplace factor $n^{-3/2}$ times $e^{n\Phi(t_*,t_*,t_*)}$. Strict concavity makes the integral over the complement exponentially smaller. Meanwhile, Stirling's formula gives
$$P(n) = (2\pi)^{-3/2}2^{9/2}n^{3/2}64^n(1+O(n^{-1})).$$
The factors $n^{3/2}$ and $n^{-3/2}$ cancel. Using the saddle equation,
$$1 + qt_*^3 = \frac{1-t_*}{t_*},$$
we obtain
$$f_n = c_\pi \mu_\pi^n(1+O(n^{-1})), \qquad c_\pi > 0. \tag{78}$$
The third statement follows from the second. $\square$

Lemma 32 (Isolated smallest root)**.** $\chi_\pi(\mu_\pi) = 0$, $0 < \mu_\pi < 1$, *and* μ_π *is the unique root of* χ_π *in* $|\lambda| < 2$*; the other three roots have modulus greater than* 2.

Proof. Exact elimination gives
$$\operatorname{monic}\big(\operatorname{Res}_t\left(1-2t-qt^4, \lambda - 64t^4(1-t)^2\right)\big) = \chi_\pi(\lambda). \tag{79}$$
Thus $\chi_\pi(\mu_\pi) = 0$. Since $0 < t_* < 1/2$ and $64t^4(1-t)^2$ is increasing on $(0, 1/2)$, one has $0 < \mu_\pi < 1$. Writing
$$\chi_\pi(\lambda) = c_4\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0,$$
exact rational arithmetic at $q = 53360^{-3}$ proves
$$2|c_1| > |c_0| + 4|c_2| + 8|c_3| + 16|c_4|. \tag{80}$$
On $|\lambda| = 2$, the term $c_1\lambda$ therefore dominates the sum of all other terms. Rouché's theorem shows that χ_π has exactly one root in $|\lambda| < 2$, namely μ_π; the other three roots have modulus greater than 2. $\square$

Lemma 33 (Recessiveness)**.** *f_n is the unique recessive companion solution.*

Proof. By Lemma 31 $f_n^{1/n} \to \mu_\pi$, and by Lemma 32 μ_π is the unique root of smallest modulus; Lemma 2 gives the claim. ☐

Lemma 34 (Column connections)**.** *All four columns connect to the line.*

Proof. For a row companion matrix with limiting last column $(c_0, c_1, c_2, c_3)^\mathsf{T}$, put

$$r_C(\lambda) = \begin{pmatrix} \lambda^3 - c_3\lambda^2 - c_2\lambda - c_1 \\ \lambda^2 - c_3\lambda - c_2 \\ \lambda - c_3 \\ 1 \end{pmatrix}.$$

Then $r_C(\mu_\pi)^\mathsf{T}$ is the limiting left eigenvector for the recessive mode of $C_\pi(n)^\mathsf{T}$. If $h_j(\lambda)$ is the numerator of the jth entry of $B_\pi r_C(\lambda)$ after cancellation, the notebook verifies

$$\gcd(\chi_\pi, h_j) = 1 \qquad (j = 0, 1, 2, 3). \tag{81}$$

Since $\chi_\pi(\mu_\pi) = 0$, every $h_j(\mu_\pi)$ is nonzero. Thus all four standard columns satisfy the connection hypothesis of Corollary 1. ☐

11.4 The companion initial vector and the Chudnovsky functionals

Set

$$\mathbf{f}_{\pi,0} = (f_0, f_1, f_2, f_3)^\mathsf{T}.$$

By construction,

$$\mathbf{f}_{\pi,0} = K_\pi(0)^\mathsf{T} Y_0^\mathsf{T}, \qquad Y_0^\mathsf{T} = K_\pi(0)^{-\mathsf{T}} \mathbf{f}_{\pi,0}. \tag{82}$$

We now evaluate the two linear functionals required by the question directly from the classical Chudnovsky identity.

Let

$$A_0 = 545140134, \qquad B_0 = 13591409, \qquad C_0 = 426880,$$

$$c_k = \frac{(1/6)_k (1/2)_k (5/6)_k}{(k!)^3}.$$

We use the classical Chudnovsky formula [42]

$$G_0 := \sum_{k=0}^{\infty} (A_0 k + B_0) c_k z_0^k = C_0 \frac{\sqrt{10005}}{\pi}. \tag{83}$$

For $N \geq 0$, define the normalized weighted tail

$$G_N = \frac{1}{c_N z_0^N} \sum_{k=N}^{\infty} (A_0 k + B_0) c_k z_0^k$$

and

$$F_N(z) = {}_4F_3\left(\begin{matrix} 1, N + \frac{1}{6}, N + \frac{1}{2}, N + \frac{5}{6} \\ N+1, N+1, N+1 \end{matrix} ; z \right).$$

Coefficient comparison gives the two elementary tail identities

$$G_N = (A_0 N + B_0) F_N(z_0) + A_0 \Theta F_N(z_0),$$

$$F_N(z_0) - \rho_{N+1} F_{N+1}(z_0) = 1, \qquad \rho_{N+1} = z_0 \frac{(N + 1/6)(N + 1/2)(N + 5/6)}{(N+1)^3}.$$

Let T_{12} be the exact contiguous transport from

$$\left(1, \frac{7}{6}, \frac{3}{2}, \frac{11}{6}; 2, 2, 2\right)$$

to

$$\left(1, \frac{13}{6}, \frac{5}{2}, \frac{17}{6}; 3, 3, 3\right),$$

let $e_1 = (1,0,0,0)^{\mathsf{T}}$, and define

$$r = e_1 - \rho_2 T_{12} e_1, \qquad u_1 = (A_0 + B_0, A_0, 0, 0)^{\mathsf{T}},$$

$$\alpha = B_0 r + \rho_1 u_1, \qquad \beta = C_0 r. \tag{84}$$

The tail identities give

$$Y_0 \alpha = G_0, \qquad Y_0 \beta = C_0. \tag{85}$$

Thus

$$IC_{\text{raw}} = \begin{pmatrix} \alpha^{\mathsf{T}} \\ \beta^{\mathsf{T}} \end{pmatrix}$$

is derived from the scalar Chudnovsky series rather than guessed. Exact rational simplification gives the relation with the initial matrix in the question:

$$A = 2734764715008000\, IC_{\text{raw}}. \tag{86}$$

Combining (82), (85), and (86) yields the companion-coordinate evaluation

$$AK_\pi(0)^{-\mathsf{T}} \mathbf{f}_{\pi,0} = 2734764715008000 \cdot 426880 \begin{pmatrix} \frac{\sqrt{10005}}{\pi} \\ 1 \end{pmatrix}. \tag{87}$$

This evaluates exactly the two linear functionals needed for the challenge; no separate closed forms for the four numbers f_0, f_1, f_2, f_3 are required.

Theorem 8 (Approximation of $\sqrt{10005}/\pi$)**.** *For each* $j = 1,2,3,4$, $\lim\limits_{N\to\infty} \dfrac{P_{N,j}}{Q_{N,j}} = \dfrac{\sqrt{10005}}{\pi}$.

Proof. By Corollary 2, Lemma 34, and Lemma 29, for every $j = 1,2,3,4$ there are nonzero scalars $\gamma_{N,j}$ with

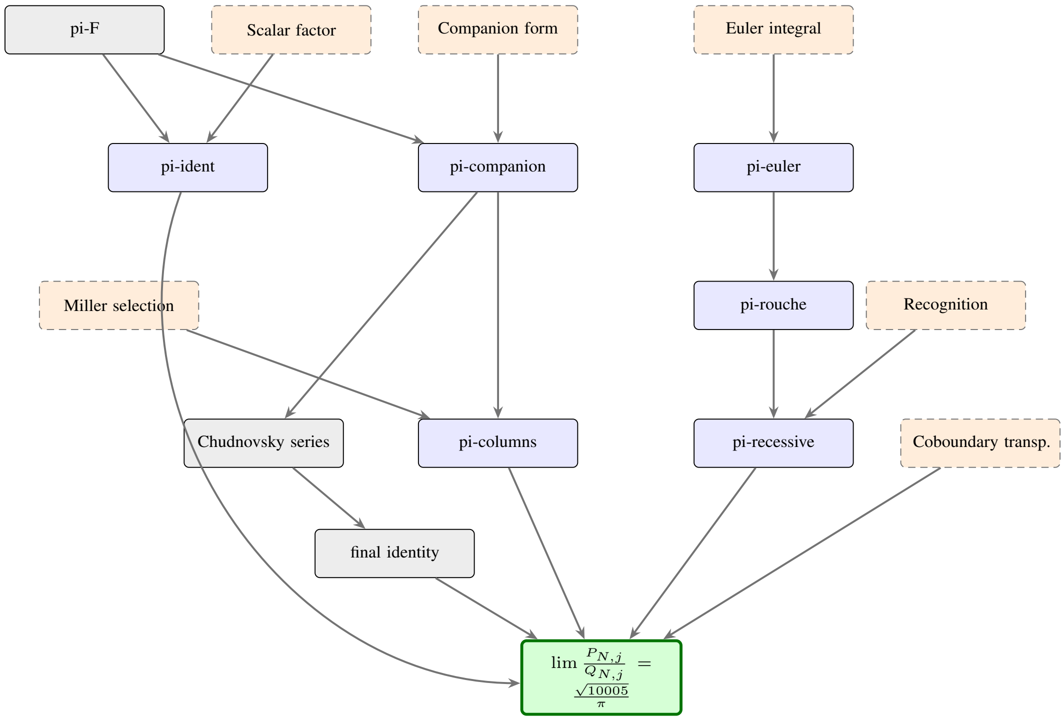
$$A\mathcal{M}_N e_j = \gamma_{N,j} \left(2734764715008000 \cdot 426880 \begin{pmatrix} \sqrt{10005}/\pi \\ 1 \end{pmatrix} + o(1)\right),$$

where the limiting vector is obtained above. The second component is nonzero for all sufficiently large N, so $P_{N,j}/Q_{N,j} \to \sqrt{10005}/\pi$. $\square$

11.5 Connection with code

The notebook constructs the `1111222` trajectory, verifies all sixteen entries of (72), constructs the companion coboundary (73) and checks its determinant, computes (74), the resultant (79), the Rouché inequality (80), and the four gcd certificates (81). It derives T_{12}, verifies (84), proves the scaling (86), and checks (87).

11.6 Blueprint dependency graph



12 Acknowledgments

This research received support through Schmidt Sciences, LLC.

13 Statement on use of AI

AI has been used in the generation of this work. Notably:

- Some code was written and revised completely by Codex (GPT 5.6 Sol/High), based on existing custom libraries. It was reviewed by the authors and Gemini 3.6 Thinking.
- Proofs were adapted from [22] to other specifications by GPT 5.6 Sol/Pro. They were then checked by GPT 5.6 Sol/High, Gemini 3.1 Pro, Grok 4.5 Fast, and Claude Opus 4.8/High (as well as manually by the authors).
- All dependency graphs were generated by Claude Opus 4.8/Extra.
- Claude Opus 4.8/Extra was used to reformat and standardize the different writing styles of the authors.

References

[1] Jie Zhang et al. “RealMath: A Continuous Benchmark for Evaluating Language Models on Research-Level Mathematics”. In: *arXiv preprint arXiv:2505.12575* (2025). DOI: `10.48550/arXiv.2505.12575`. URL: `https://arxiv.org/abs/2505.12575`.

[2] Antoine Peyronnet, Fabian Gloeckle, and Amaury Hayat. *LemmaBench: A Live, Research-Level Benchmark to Evaluate LLM Capabilities in Mathematics*. 2026. arXiv: `2602.24173 [cs.AI]`. URL: `https://arxiv.org/abs/2602.24173`.

[3] Seyed Iman Mirzadeh et al. “GSM-Symbolic: Understanding the Limitations of Mathematical Reasoning in Large Language Models”. In: *The Thirteenth International Conference on Learning Representations*. 2025. URL: `https://openreview.net/forum?id=AjXkRZIvjB`.

[4] Michael Shalyt, Rotem Elimelech, and Ido Kaminer. *ASyMOB: Algebraic Symbolic Mathematical Operations Benchmark*. 2025. arXiv: 2505.23851 [cs.CL]. URL: https://arxiv.org/abs/2505.23851.

[5] Elliot Glazer, Ege Erdil, Tamay Besiroglu, et al. "FrontierMath: A Benchmark for Evaluating Advanced Mathematical Reasoning in AI". In: *arXiv preprint arXiv:2411.04872* (2024). DOI: 10.48550/arXiv.2411.04872. URL: https://arxiv.org/abs/2411.04872.

[6] Suhaas Garre et al. *Riemann-Bench: A Benchmark for Moonshot Mathematics*. 2026. arXiv: 2604.06802 [cs.AI]. URL: https://arxiv.org/abs/2604.06802.

[7] Long Phan, ...(others)..., and Dan Hendrycks. *Humanity's Last Exam*. 2025. arXiv: 2501.14249 [cs.LG]. URL: https://arxiv.org/abs/2501.14249.

[8] Mohammed Abouzaid, Andrew J. Blumberg, Martin Hairer, et al. "First Proof". In: *arXiv preprint arXiv:2602.05192* (2026). DOI: 10.48550/arXiv.2602.05192. URL: https://arxiv.org/abs/2602.05192.

[9] Srinivasa Ramanujan. "Modular Equations and Approximations to π". In: *The Quarterly Journal of Pure and Applied Mathematics* 45 (1914), pp. 350–372.

[10] Roger Apéry. "Irrationalité de $\zeta(2)$ et $\zeta(3)$". In: *Journées Arithmétiques de Luminy*. Vol. 61. Astérisque. Société Mathématique de France, 1979, pp. 11–13.

[11] Frits Beukers. "A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$". In: *Bulletin of the London Mathematical Society* 11.3 (1979), pp. 268–272. DOI: 10.1112/blms/11.3.268.

[12] Keith Ball and Tanguy Rivoal. "Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs". In: *Inventiones Mathematicae* 146.1 (2001), pp. 193–207. DOI: 10.1007/s002220100168.

[13] Wadim Zudilin. "One of the Numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is Irrational". In: *Russian Mathematical Surveys* 56.4 (2001), pp. 774–776. DOI: 10.1070/RM2001v056n04ABEH000427.

[14] Tanguy Rivoal and Wadim Zudilin. "Diophantine Properties of Numbers Related to Catalan's Constant". In: *Mathematische Annalen* 326.4 (2003), pp. 705–721. DOI: 10.1007/s00208-003-0420-2.

[15] Jeffrey C. Lagarias. "Euler's Constant: Euler's Work and Modern Developments". In: *Bulletin of the American Mathematical Society* 50.4 (2013), pp. 527–628. DOI: 10.1090/S0273-0979-2013-01423-X.

[16] Marc Chamberland and Armin Straub. "Apéry Limits: Experiments and Proofs". In: *American Mathematical Monthly* 128.9 (2021), pp. 811–824. DOI: 10.1080/00029890.2021.1958005.

[17] Gal Raayoni et al. "Generating conjectures on fundamental constants with the Ramanujan Machine". In: *Nature* 590.7844 (2021), pp. 67–73.

[18] Rotem Elimelech et al. "Algorithm-assisted discovery of an intrinsic order among mathematical constants". In: *Proceedings of the National Academy of Sciences* 121.25 (2024), e2321440121. DOI: 10.1073/pnas.2321440121. eprint: https://www.pnas.org/doi/pdf/10.1073/pnas.2321440121. URL: https://www.pnas.org/doi/abs/10.1073/pnas.2321440121.

[19] Shachar Weinbaum et al. *On Conservative Matrix Fields: Continuous Asymptotics and Arithmetic*. 2025. arXiv: 2507.08138 [math.NT]. URL: https://arxiv.org/abs/2507.08138.

[20] Maxim Kontsevich and Don Zagier. "Periods". In: *Mathematics Unlimited—2001 and Beyond*. Springer, 2001, pp. 771–808.

[21] Ramanujan Machine Group. *ramanujantools*. 2025. DOI: 10.5281/ZENODO.15747099. URL: https://zenodo.org/doi/10.5281/zenodo.15747099.

[22] Tomer Raz et al. "From Euler to AI: Unifying Formulas for Mathematical Constants". In: *Advances in Neural Information Processing Systems*. Ed. by D. Belgrave et al. Vol. 38. Curran Associates, Inc., 2025, pp. 123959–124040. URL: https://proceedings.neurips.cc/paper_files/paper/2025/file/b394adff21327520268aaf565b63414c-Paper-Conference.pdf.

[23] George E. Andrews, Richard Askey, and Ranjan Roy. *Special Functions*. Cambridge University Press, 1999.

[24] Alin Bostan, Frédéric Chyzak, and Élie De Panafieu. "Complexity estimates for two uncoupling algorithms". In: *Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation*. 2013, pp. 85–92.

[25] Saber Elaydi. *An Introduction to Difference Equations*. Springer, 2005.

[26] *NIST Digital Library of Mathematical Functions*. https://dlmf.nist.gov/, Release 1.2.4 of 2025-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. URL: https://dlmf.nist.gov/.

[27] Walter Gautschi. "Computational Aspects of Three-Term Recurrence Relations". In: *SIAM Review* 9.1 (1967), pp. 24–82. DOI: 10.1137/1009002.

[28] Ray VM Zahar. "A mathematical analysis of Miller's algorithm". In: *Numerische Mathematik* 27.4 (1976), pp. 427–447.

[29] George D. Birkhoff and Waldemar J. Trjitzinsky. "Analytic Theory of Singular Difference Equations". In: *Acta Mathematica* 60.1 (1933), pp. 1–89.

[30] Manuel Kauers and Peter Paule. *The Concrete Tetrahedron: Symbolic Sums, Recurrence Equations, Generating Functions, Asymptotic Estimates*. Springer, 2011. DOI: 10.1007/978-3-7091-0445-3.

[31] Manuel Kauers. *A Mathematica package for computing asymptotic expansions of solutions of P-finite recurrence equations*. RISC Report Series 11-04. ISSN 2791-4267. Linz, Austria: Research Institute for Symbolic Computation (RISC), Johannes Kepler University, 2011.

[32] Carsten Schneider. "Symbolic Summation Assists Combinatorics". In: *Séminaire Lotharingien de Combinatoire* 56 (2007). Article B56b, pp. 1–36. URL: https://www.mat.univie.ac.at/~slc/wpapers/s56schneider.html.

[33] Jakob Ablinger. "The Package HarmonicSums: Computer Algebra and Analytic Aspects of Nested Sums". In: *Proceedings of Science* LL2014 (2014), p. 019. DOI: 10.22323/1.211.0019. URL: https://doi.org/10.22323/1.211.0019.

[34] Carsten Schneider. "Modern Summation Methods for Loop Integrals in Quantum Field Theory: The Packages Sigma, EvaluateMultiSums and SumProduction". In: *Journal of Physics: Conference Series* 523.1 (2014), p. 012037. DOI: 10.1088/1742-6596/523/1/012037. URL: https://doi.org/10.1088/1742-6596/523/1/012037.

[35] Carsten Schneider. "Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation". In: *Anti-Differentiation and the Calculation of Feynman Amplitudes*. Ed. by Johannes Blümlein and Carsten Schneider. Texts and Monographs in Symbolic Computation. Cham: Springer, 2021, pp. 423–485. DOI: 10.1007/978-3-030-80219-6_17. URL: https://doi.org/10.1007/978-3-030-80219-6_17.

[36] Doron Zeilberger. "The Method of Creative Telescoping". In: *Journal of Symbolic Computation* 11.3 (1991), pp. 195–204. DOI: 10.1016/S0747-7171(08)80044-2. URL: https://doi.org/10.1016/S0747-7171(08)80044-2.

[37] Jakob Ablinger, Johannes Blümlein, and Carsten Schneider. "Analytic and Algorithmic Aspects of Generalized Harmonic Sums and Polylogarithms". In: *Journal of Mathematical Physics* 54.8 (2013), p. 082301. DOI: 10.1063/1.4811117. URL: https://doi.org/10.1063/1.4811117.

[38] Jakob Ablinger. "Discovering and Proving Infinite Binomial Sums Identities". In: *Experimental Mathematics* 26.1 (2017), pp. 62–71. DOI: 10.1080/10586458.2015.1116028. URL: https://doi.org/10.1080/10586458.2015.1116028.

[39] Jakob Ablinger et al. "Iterated Binomial Sums and Their Associated Iterated Integrals". In: *Journal of Mathematical Physics* 55.11 (2014), p. 112301. DOI: 10.1063/1.4900836. URL: https://doi.org/10.1063/1.4900836.

[40] Jakob Ablinger, Johannes Blümlein, and Carsten Schneider. "Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials". In: *Journal of Mathematical Physics* 52.10 (2011), p. 102301. DOI: 10.1063/1.3629472. URL: https://doi.org/10.1063/1.3629472.

[41] Jakob Ablinger. “Proving Two Conjectural Series for $\zeta(7)$ and Discovering More Series for $\zeta(7)$”. In: *Mathematical Aspects of Computer and Information Sciences*. Ed. by Daniel Slamanig, Elias Tsigaridas, and Zafeirakis Zafeirakopoulos. Vol. 11989. Lecture Notes in Computer Science. Cham: Springer, 2020, pp. 42–47. DOI: 10.1007/978-3-030-43120-4_5. URL: https://doi.org/10.1007/978-3-030-43120-4_5.

[42] David V. Chudnovsky and Gregory V. Chudnovsky. “Approximations and Complex Multiplication According to Ramanujan”. In: *Ramanujan Revisited*. Boston: Academic Press, 1988, pp. 375–472.